Test 3 Review

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Test 3

- Test 3: Dec. 4-6 in CASA
- Material - Through 6.3.

No Homework (Thanksgiving)

- No homework this week!
- Have a GREAT Thanksgiving!

Final Exam

- Final Exam: Dec. 14-17 in CASA

You Might Be Interested to Know ...

- I will replace your lowest test score with the percentage grade from the final exam (provided it is higher).
- I will give an A to anyone who receives 95% or above on the final exam.
- I will give a passing grade to anyone who receives at least 70% on the final exam.

Quiz 1

What is today?

a. Monday

b. Wednesday

c. Friday

d. None of these

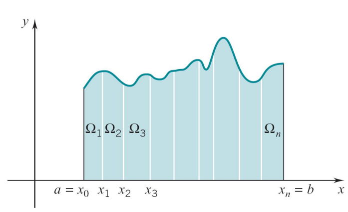
Test 3 Material

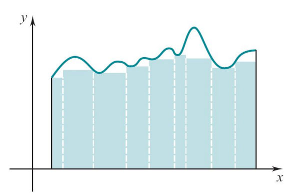
- Test 3 will cover material from Chapter 5, along with Sections 6.1, 6.2 and 6.3.

Good Sources of Practice Problems

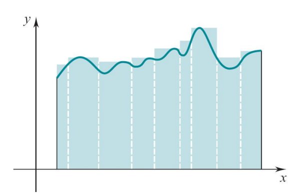
- Examples from class.
- The basic homework problems.
- The basic online quiz problems.

Definite Integral and Lower/Upper Sums





area of shaded region is a lower sum for f



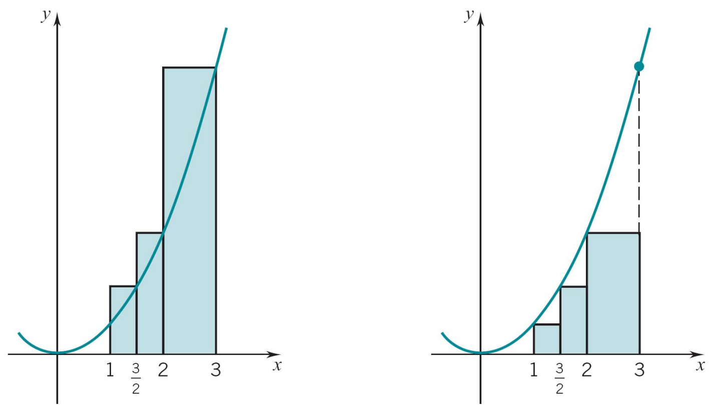
area of shaded region is an upper sum for f

$$\begin{aligned}\text{Area of } \Omega = &\ \text{Area of } \Omega_1 + \text{ Area of } \Omega_2 + \cdots + \text{ Area of } \Omega_n,\\ L_f(P) &= m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n\\ U_f(P) &= M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n\end{aligned}$$

$$L_f(P) \le \int_a^b f(x)\,dx \le U_f(P), \qquad \text{for all partitions } P \text{ of } [a,b]$$

Problem 1

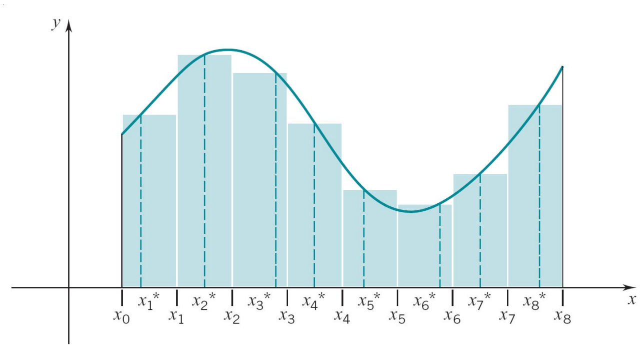
Give both the upper and lower Riemann sums for the function $f(x) = x^2$ over the interval $[1, 3]$ with respect to the partition $P = \{1, \frac{3}{2}, 2, 3\}$.



Upper sum

Lower sum

Lower/Upper Sums and Riemann Sums

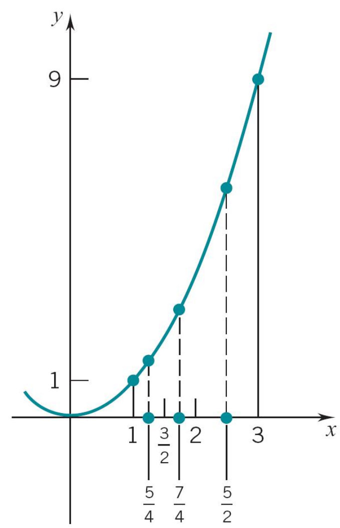


$$S^*(P) = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

$$L_f(P) \le S^*(P) \le U_f(P), \quad \text{for all partitions } P \text{ of } [a, b]$$

Problem 2

Give the Riemann sums for the function $f(x) = x^2$ over the interval $[1, 3]$ with respect to the partition $P = \{1, \frac{3}{2}, 2, 3\}$ using midpoints.



Fundamental Theorem of Integral Calculus

Theorem 1. *In general,*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

where $F(x)$ is an antiderivative of $f(x)$.

Function	Antiderivative
x^r	$\dfrac{x^{r+1}}{r+1}$ (r a rational number $\neq -1$)
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc^2 x$	$-\cot x$
$\csc x \cot x$	$-\csc x$

Problem 3

Evaluate

1. $\displaystyle\int_0^1 \left(2x - 6x^4 + 5\right) dx$

2. $\displaystyle\int_1^2 \frac{x^4 + 1}{x^2}\, dx$

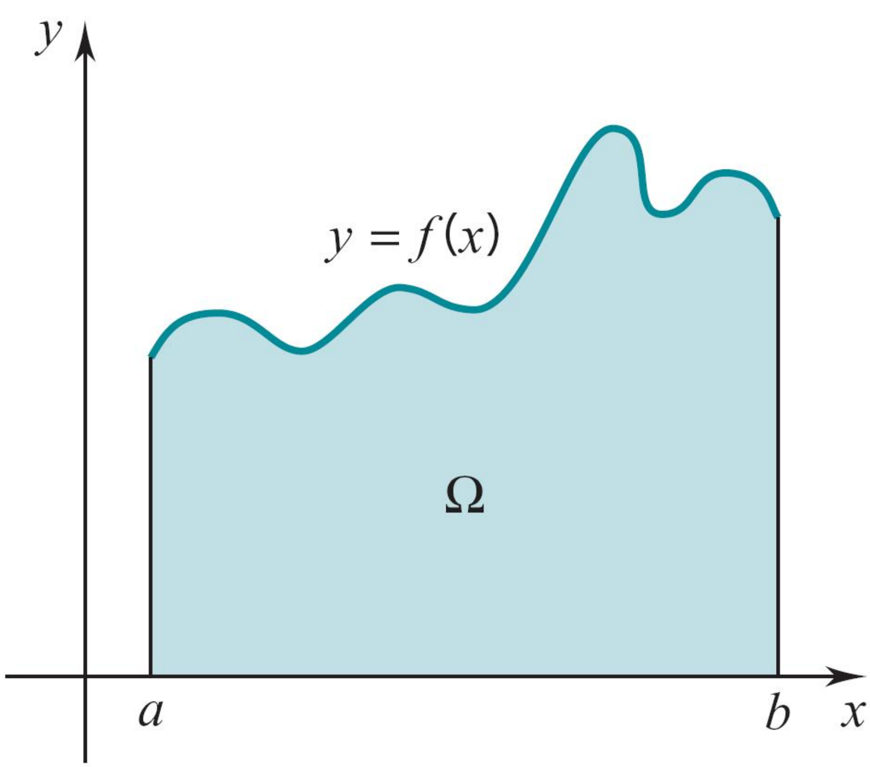
3. $\displaystyle\int_0^1 \left(4 - \sqrt{x}\right)^2 dx$

4. $\displaystyle\int_0^{\pi/4} \sec x\left(2\tan x - 5\sec x\right) dx$

Area below the graph of a Nonnegative f

$$f(x) \geq 0 \quad \text{for all } x \text{ in } [a, b].$$

$$\Omega = \text{ region below the graph of } f.$$

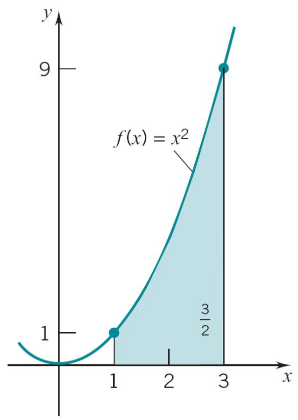


$$\text{Area of } \Omega = \int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

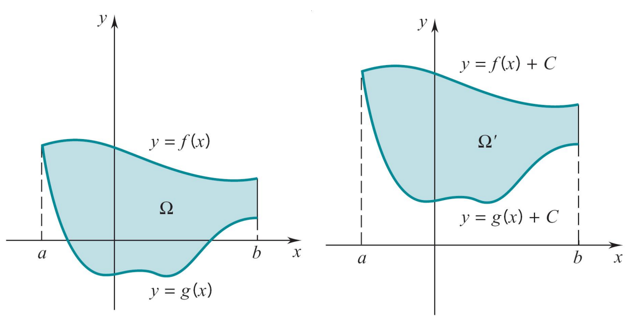
Problem 4

Find the area bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 3]$.



Area of the shaded region: $\int_1^3 x^2\,dx = \frac{26}{3}$

Area between the graphs of f and g

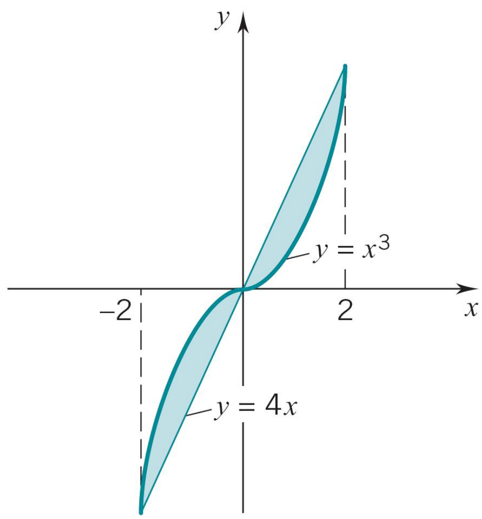


$f(x) \geq g(x)$ for all x in $[a, b]$.

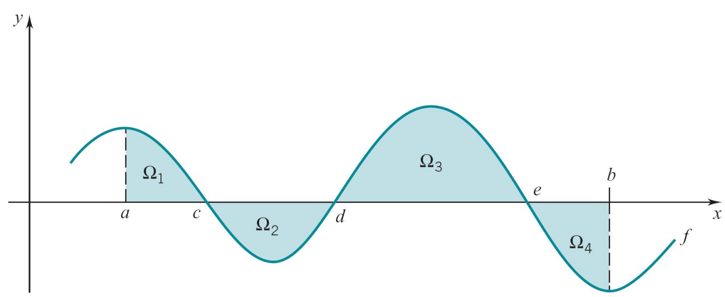
$\Omega =$ region between the graphs of f (Top) and g (Bottom).

$$\text{Area of } \Omega = \int_a^b [\text{ Top } - \text{ Bottom }]\,dx = \int_a^b [f(x) - g(x)]\,dx.$$

Problem 5
Find the area between the graphs of $y = 4x$ and $y = x^3$ over the interval $[-2, 2]$.



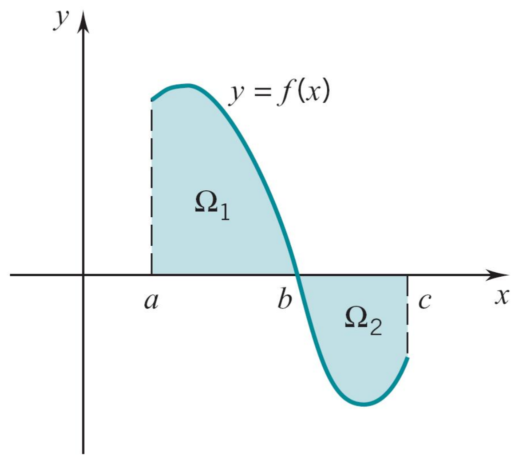
$\int_a^b f(x)\,dx$ **as Signed Area**



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^e f(x)\,dx + \int_e^b f(x)\,dx$$
$$= \text{Area of } \Omega_1 - \text{Area of } \Omega_2 + \text{Area of } \Omega_3 - \text{Area of } \Omega_4$$
$$= \left[\text{Area of } \Omega_1 + \text{Area of } \Omega_3\right] - \left[\text{Area of } \Omega_2 + \text{Area of } \Omega_4\right]$$
$$= \text{Area above the } x\text{-axis} - \text{Area below the } x\text{-axis.}$$

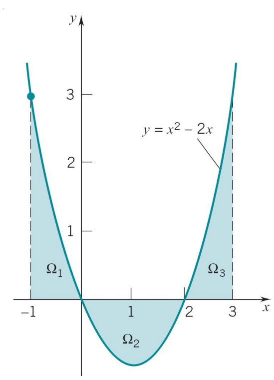
Problem 6

The graph of $y = f(x)$ is shown below. The region Ω_2 has area 3 and $\displaystyle\int_a^c f(x)dx$ is 2. Give the area of region Ω_1.



Problem 7

- Give the area bounded between the graph of $f(x) = x^2 - 2x$ and the x-axis on $[-1, 3]$.
- Evaluate $\int_{-1}^{3} (x^2 - 2x)\, dx$ and interpret the result in terms of areas.



Indefinite Integral as General Antiderivative

The Indefinite Integral of f
In general,

$$\int f(x)\, dx = F(x) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and C is an arbitrary constant.

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C \qquad (r \text{ rational, } r \neq -1)$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc^2 x \, dx = -\cot x + C \qquad \int \csc x \cot x \, dx = -\csc x + C$$

Problem 8

1. Find F given that $F'(x) = \cos 3x$ and $F(-\pi) = 1$.

2. Give an antiderivative of $f(x) = \cos 3x$ whose graph has y-intercept 3.

Undoing the Chain Rule: The u-Substitution

Note: Most differentiation involves the chain rule, so we should expect that most antidifferentiation will involve

undoing the chain rule [1ex] (*the u-Substitution*)

If F is an antiderivative for f, then

$$\frac{d}{dx}\big[F(u(x))\big] = F'(u(x))\, u'(x) = f(u(x))\, u'(x)$$

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du = F(u) + C = F(u(x)) + C.$$

The u-Substitution

Original Integral	$u = g(x)$, $du = g'(x)dx$	New Integral
$\int [g(x)]^r g'(x)\,dx$	$\rightarrow$	$\int u^r du = \frac{u^{r+1}}{r+1} + C = \frac{[g(x)]^{r+1}}{r+1} + C \quad (r \neq -1)$
$\int \sin[g(x)]\,g'(x)\,dx$	$\rightarrow$	$\int \sin u\,du = -\cos u + C = -\cos[g(x)] + C$
$\int \cos[g(x)]\,g'(x)\,dx$	$\rightarrow$	$\int \cos u\,du = \sin u + C = \sin[g(x)] + C$
$\int \sec^2[g(x)]\,g'(x)\,dx$	$\rightarrow$	$\int \sec^2 u\,du = \tan u + C = \tan[g(x)] + C$
$\int \sec[g(x)]\tan[g(x)]g'(x)\,dx$	$\rightarrow$	$\int \sec u\,\tan u\,du = \sec u + C = \sec[g(x)] + C$
$\int \csc^2[g(x)]\,g'(x)\,dx$	$\rightarrow$	$\int \csc^2 u\,du = -\cot u + C = -\cot[g(x)] + C$
$\int \csc[g(x)]\cot[g(x)]\,g'(x)\,dx$	$\rightarrow$	$\int \csc u \cot u\,du = -\csc u + C = -\csc[g(x)] + C$

Problem 9
Calculate

1. $\int \sin x \cos x \, dx$
2. $\int 2x^3 \sec^2(x^4+1)\, dx$
3. $\int \sec^3 x \tan x \, dx$
4. $\int x(x-3)^5 \, dx$

Substitution in Definite Integrals

The Change of Variables Formula

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

We change the limits of integration to reflect the substitution.

Problem 10

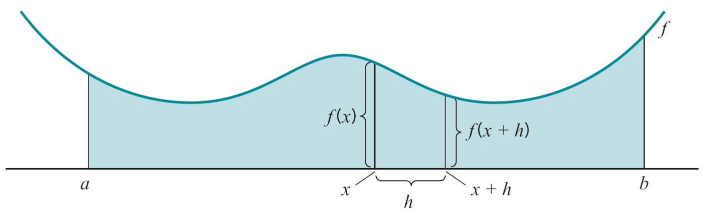
Evaluate

$$1.\quad \int_0^2 (x^2-1)(x^3-3x+2)^3\,dx$$

$$2.\quad \int_0^{1/2} \cos^3 \pi x \sin \pi x\,dx$$

$$3.\quad \int_0^{\sqrt{3}} x^5\sqrt{x^2+1}\,dx$$

Definite Integral and Antiderivative



$F(x)$ = area from a to x and $F(x+h)$ = area from a to $x+h$.
Therefore $F(x+h)-F(x)$ = area from x to $x+h \cong f(x)\,h$ if h is small and

$$\frac{F(x+h)-F(x)}{h} \cong \frac{f(x)\,h}{h} = f(x).$$

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Problem 11

1. Find $f(x)$ such that $\displaystyle\int_{-2}^{x} f(t)\,dt = \cos(2x)+1$.

2. Give the function $f(x)$ that solves the equation $\displaystyle\int_2^x (t+1)f(t)\,dt = \sin(x)$.

Properties

$$\frac{d}{dx}\left(\int_a^u f(t)\,dt\right) = f(u)\frac{du}{dx}$$

$$\frac{d}{dx}\left(\int_v^b f(t)\,dt\right) = -f(v)\frac{dv}{dx}$$

$$\frac{d}{dx}\left(\int_v^u f(t)\,dt\right) = f(u)\frac{du}{dx} - f(v)\frac{dv}{dx}$$

Problem 12

Find

1. $$\frac{d}{dx}\left(\int_0^{x^3} \frac{dt}{1+t}\right)$$

2. $$\frac{d}{dx}\left(\int_{-3}^{x^2} \left(3t - \sin(t^2)\right) dt\right)$$

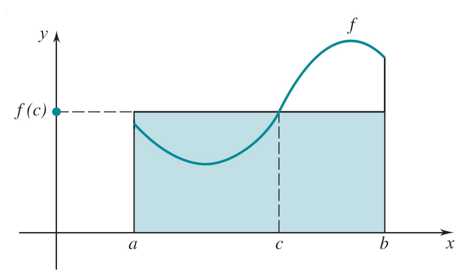
3. $$\frac{d}{dx}\left(\int_x^{2x} \frac{dt}{1+t^2}\right)$$

Mean-Value Theorems for Integrals
Let f_{avg} denote the average or mean value of f on $[a, b]$. Then

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The First Mean-Value Theorems for Integrals
If f is continous on $[a, b]$, then there is at least one number c in (a, b) for which

$$f(c) = f_{\text{avg}}.$$



Problem 13

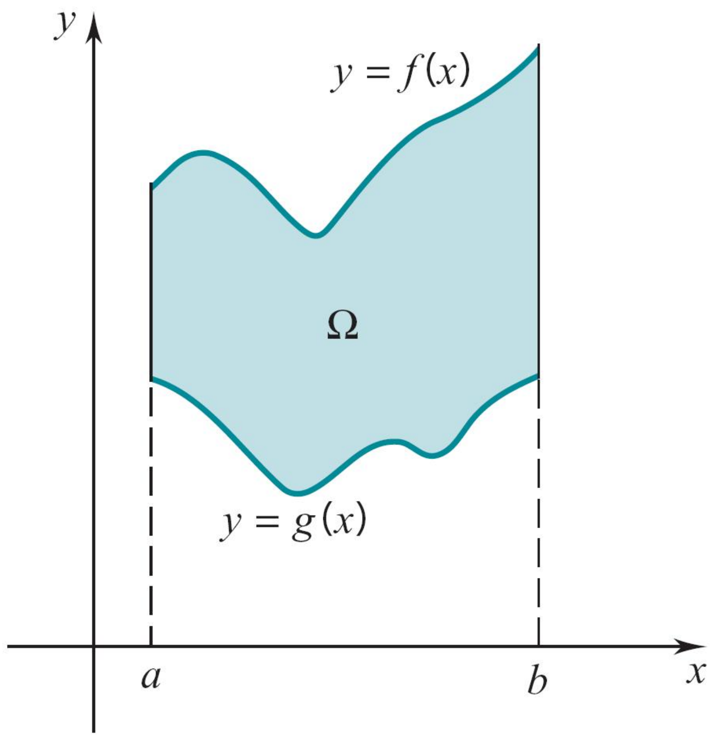
Give the average value of the function $f(x) = \sin x$ on the interval $[0, \pi/2]$.

Problem 14

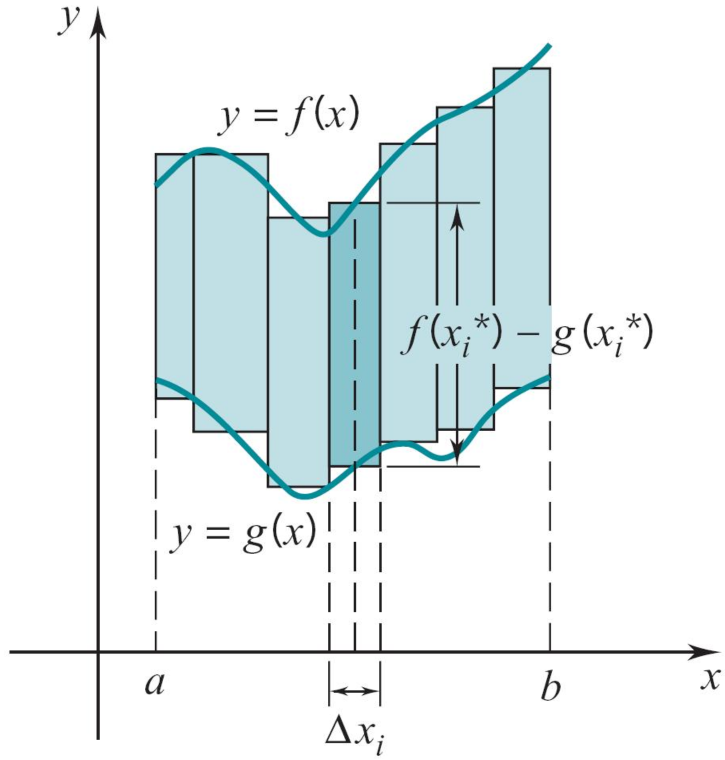
Give the value of c that satisfies the conclusion of the mean value theorem for integrals for the function $f(x) = x^2 - 2x + 3$ on the interval $[1, 4]$.

Area by Integration with Respect to x: $f(x) \geq g(x)$

Rectangle Area $[f(x_i^*) - g(x_i^*)]\Delta x_i$



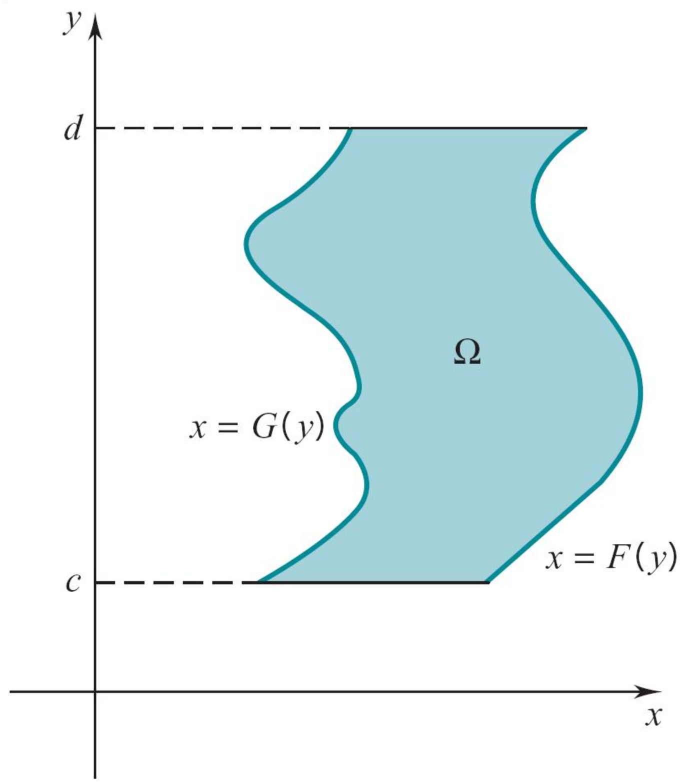
Riemann Sum $\sum [f(x_i^*) - g(x_i^*)]\Delta x_i$



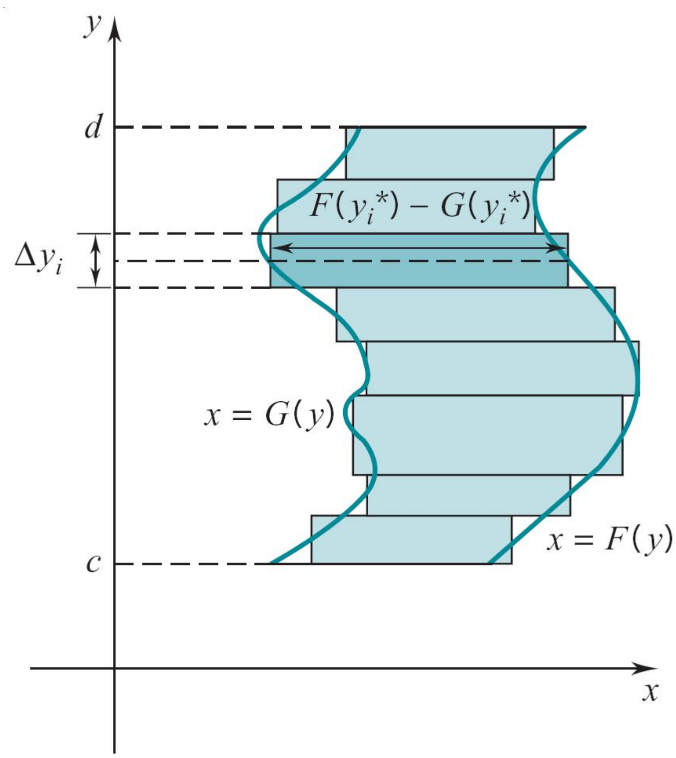
$$\text{area}(\Omega) = \int_a^b [f(x) - g(x)]\, dx = \lim_{\|P\| \to 0} \sum [f(x_i^*) - g(x_i^*)] \Delta x_i.$$

Area by Integration with Respect to y: $F(y) \geq G(y)$

Rectangle Area $[F(y_i^*) - G(y_i^*)]\Delta y_i$

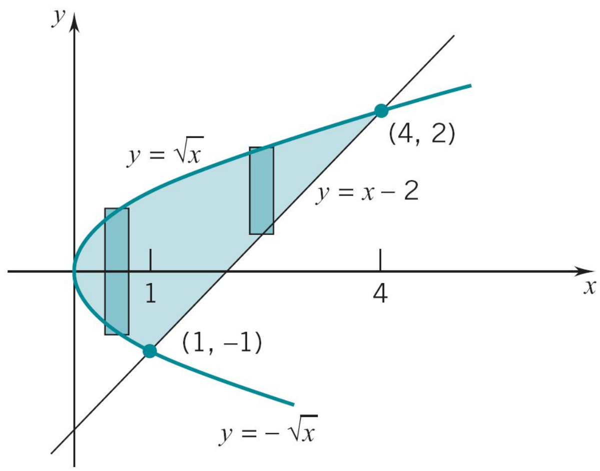


Riemann Sum $\sum[F(y_i^*) - G(y_i^*)]\Delta y_i$

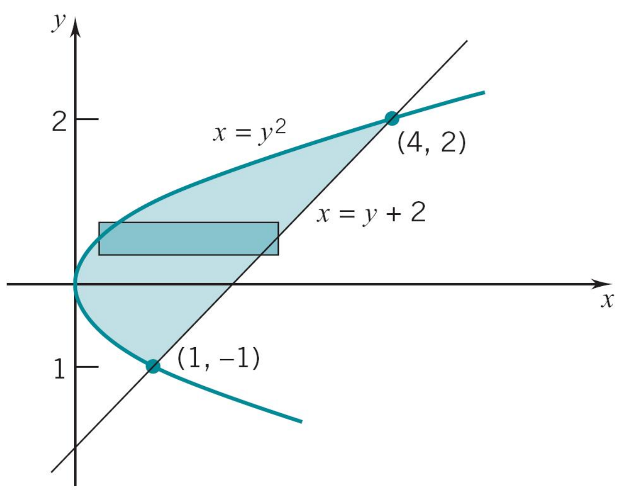


$$\text{area}(\Omega) = \int_c^d [F(y) - G(y)]\, dy = \lim_{\|P\| \to 0} \sum [F(y_i^*) - G(y_i^*)] \Delta y_i.$$

Problem 15
Give a formula involving integral(s) in x for the region bounded by $y = x - 2$ and $y = \sqrt{x}$.

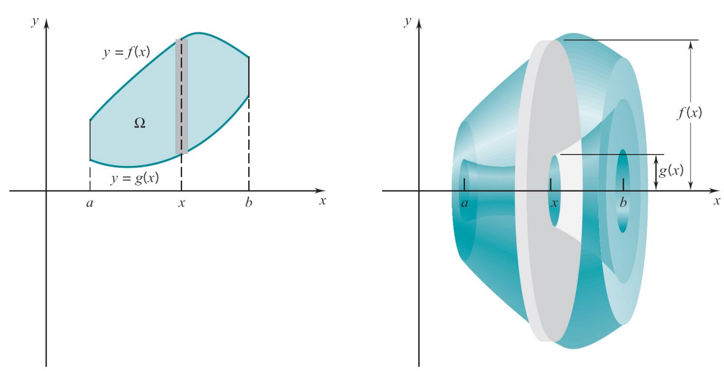


Problem 16
Give a formula involving integral(s) in y for the region bounded by $y = x - 2$ and $y = \sqrt{x}$.



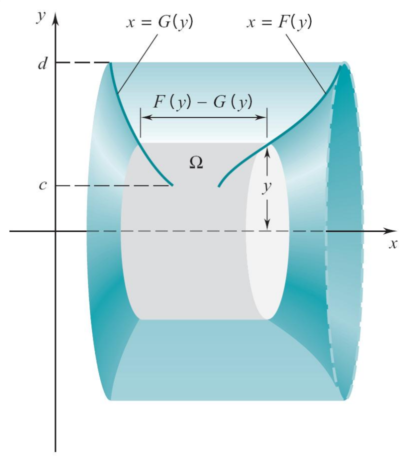
Solid of Revolution About the x-Axis: Washer

Cylinder Volume: $\pi([f(x_i^*)]^2 - [g(x_i^*)]^2)\Delta x_i$ [1ex] Riemann Sum: $\sum \pi([f(x_i^*)]^2 - [g(x_i^*)]^2)\Delta x_i$ [1ex]



$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx = \lim_{\|P\|\to 0} \sum \pi([f(x_i^*)]^2 - [g(x_i^*)]^2)\Delta x_i$$

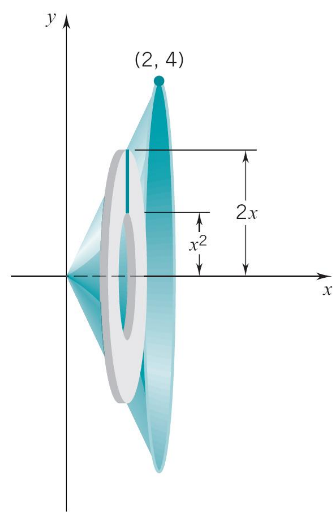
Solid of Revolution About the x-Axis: Shell



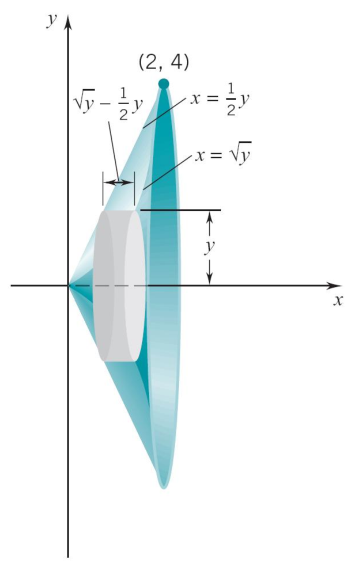
$$V = \int_c^d 2\pi\, y\,[F(y) - G(y)]\,dy = \lim_{\|P\|\to 0} \sum 2\pi\, y_i^*\,[F(y_i^*) - G(y_i^*)]\Delta y_i.$$

The integrand $2\pi\, y\,[F(y) - G(y)]$ is the lateral area of the cylinder.

Problem 17
The region bounded by $y = x^2$ and $y = 2x$ is rotated around the x-axis. Give a formula involving integrals in x for the volume of the solid that is generated.

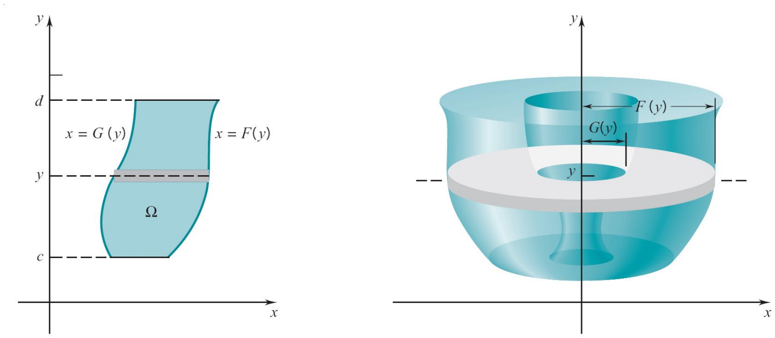


Problem 18
The region bounded by $y = x^2$ and $y = 2x$ is rotated around the x-axis. Give a formula involving integrals in y for the volume of the solid that is generated.



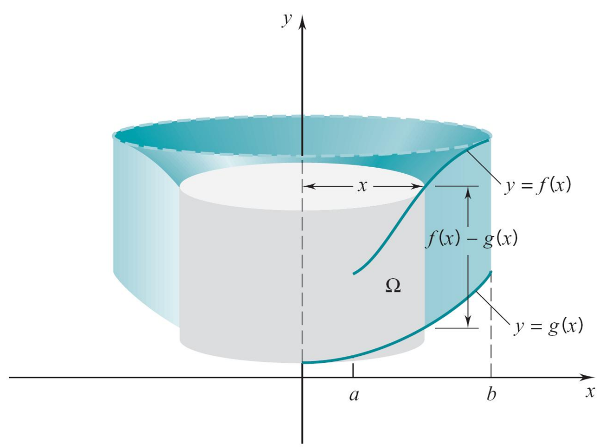
Solid of Revolution About the y-Axis: Washer

Cylinder Volume: $\pi([F(y_i^*)]^2 - [G(y_i^*)]^2)\Delta y_i$ [1ex] Riemann Sum: $\sum \pi([F(y_i^*)]^2 - [G(y_i^*)]^2)\Delta y_i$ [1ex]



$$V = \int_c^d \pi([F(y)]^2 - [G(y)]^2)\, dy = \lim_{\|P\|\to 0} \sum \pi([F(y_i^*)]^2 - [G(y_i^*)]^2)\Delta y_i$$

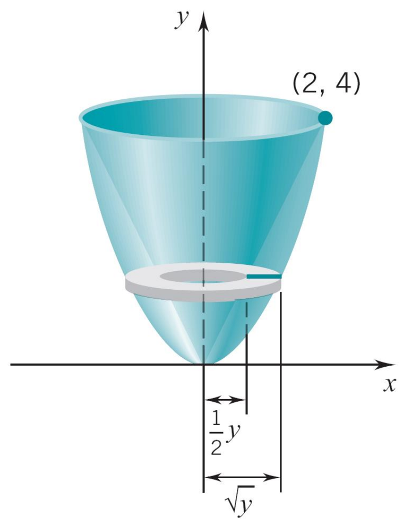
Solid of Revolution About the y-Axis: Shell



$$V = \int_a^b 2\pi\, x\, [f(x) - g(x)]\, dx = \lim_{\|P\|\to 0} \sum 2\pi\, x_i^*\, [f(x_i^*) - g(x_i^*)]\Delta x_i.$$

The integrand $2\pi\, x\, [f(x) - g(x)]$ is the lateral area of the cylinder.

Problem 19
The region bounded by $y = x^2$ and $y = 2x$ is rotated around the y-axis. Give a formula involving integrals in y for the volume of the solid that is generated.



Problem 20
The region bounded by $y = x^2$ and $y = 2x$ is rotated around the y-axis. Give a formula involving integrals in x for the volume of the solid that is generated.