Test 3 Review

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Test 3

- Test 3: Dec. 4-6 in CASA
- Material - Through 6.3.

No Homework (Thanksgiving)

- No homework this week!
- Have a GREAT Thanksgiving!

Final Exam

- Final Exam: Dec. 14-17 in CASA

You Might Be Interested to Know ...

- I will replace your lowest test score with the percentage grade from the final exam (provided it is higher).
- I will give an A to anyone who receives 95% or above on the final exam.
- I will give a passing grade to anyone who receives at least 70% on the final exam.

Quiz 1

What is today?

a. Monday
b. Wednesday
c. Friday
d. None of these
Test 3 Material

• Test 3 will cover material from Chapter 5, along with Sections 6.1, 6.2 and 6.3.

Good Sources of Practice Problems

• Examples from class.
• The basic homework problems.
• The basic online quiz problems.

Definite Integral and Lower/Upper Sums

\[
\text{Area of } \Omega = \text{Area of } \Omega_1 + \text{Area of } \Omega_2 + \cdots + \text{Area of } \Omega_n,
\]
\[
L_f(P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n
\]
\[
U_f(P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n
\]
\[
L_f(P) \leq \int_a^b f(x) \, dx \leq U_f(P), \quad \text{for all partitions } P \text{ of } [a,b]
\]

Problem 1

Give both the upper and lower Riemann sums for the function \( f(x) = x^2 \) over the interval \([1, 3]\) with respect to the partition \( P = \{1, \frac{3}{2}, 2, 3\}\).
Lower/Upper Sums and Riemann Sums

\[ S^*(P) = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \cdots + f(x_n^*) \Delta x_n \]

\[ L_f(P) \leq S^*(P) \leq U_f(P), \quad \text{for all partitions } P \text{ of } [a, b] \]

**Problem 2**

Give the Riemann sums for the function \( f(x) = x^2 \) over the interval \([1, 3]\) with respect to the partition \( P = \{1, \frac{3}{2}, 2, 3\} \) using midpoints.
Fundamental Theorem of Integral Calculus

Theorem 1. In general,

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

where \( F(x) \) is an antiderivative of \( f(x) \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^r )</td>
<td>( \frac{x^{r+1}}{r+1} ) (( r ) a rational number ( \neq -1 ))</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( \sec^2 x )</td>
<td>( \tan x )</td>
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<tr>
<td>( \sec x \tan x )</td>
<td>( \sec x )</td>
</tr>
<tr>
<td>( \csc^2 x )</td>
<td>( -\cot x )</td>
</tr>
<tr>
<td>( \csc x \cot x )</td>
<td>( -\csc x )</td>
</tr>
</tbody>
</table>

Problem 3
Evaluate

1. \( \int_0^1 (2x - 6x^4 + 5) \, dx \)

2. \( \int_1^2 \frac{x^4 + 1}{x^2} \, dx \)

3. \( \int_0^1 (4 - \sqrt{x})^2 \, dx \)

4. \( \int_0^{\pi/4} \sec x (2 \tan x - 5 \sec x) \, dx \)

Area below the graph of a Nonnegative \( f \)

\[ f(x) \geq 0 \quad \text{for all} \ x \ \text{in} \ [a, b]. \]

\( \Omega = \text{region below the graph of} \ f. \)
Area of $\Omega = \int_a^b f(x) \, dx = F(b) - F(a)$

where $F(x)$ is an antiderivative of $f(x)$.

**Problem 4**

Find the area bounded above by the graph of $f(x) = x^2$ and below by the $x$-axis over the interval $[1, 3]$.

Area between the graphs of $f$ and $g$

\[ f(x) \geq g(x) \quad \text{for all } x \in [a, b]. \]

$\Omega =$ region between the graphs of $f$ (Top) and $g$ (Bottom).

Area of $\Omega = \int_a^b \left[ \text{Top} - \text{Bottom} \right] \, dx = \int_a^b \left[ f(x) - g(x) \right] \, dx.$
Problem 5
Find the area between the graphs of $y = 4x$ and $y = x^3$ over the interval $[-2, 2]$.

\[
\int_{-2}^{2} f(x) \, dx \text{ as Signed Area}
\]

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{d} f(x) \, dx + \int_{d}^{e} f(x) \, dx + \int_{e}^{b} f(x) \, dx
\]

\[
= \text{Area of } \Omega_1 - \text{Area of } \Omega_2 + \text{Area of } \Omega_3 - \text{Area of } \Omega_4
\]

\[
= \left[ \text{Area of } \Omega_1 + \text{Area of } \Omega_3 \right] - \left[ \text{Area of } \Omega_2 + \text{Area of } \Omega_4 \right]
\]

\[
= \text{Area above the } x\text{-axis} - \text{Area below the } x\text{-axis}.
\]

Problem 6
The graph of $y = f(x)$ is shown below. The region $\Omega_2$ has area 3 and $\int_{a}^{e} f(x) \, dx$ is 2. Give the area of region $\Omega_1$. 7
Problem 7

- Give the area bounded between the graph of \( f(x) = x^2 - 2x \) and the \( x \)-axis on \([-1, 3]\).
- Evaluate \( \int_{-1}^{3} (x^2 - 2x) \, dx \) and interpret the result in terms of areas.

Indefinite Integral as General Antiderivative

The Indefinite Integral of \( f \)

In general,

\[
\int f(x) \, dx = F(x) + C
\]

where \( F(x) \) is any antiderivative of \( f(x) \) and \( C \) is an arbitrary constant.
Problem 8
1. Find $F$ given that $F'(x) = \cos 3x$ and $F(-\pi) = 1$.

2. Give an antiderivative of $f(x) = \cos 3x$ whose graph has $y$-intercept 3.

Undoing the Chain Rule: The $u$-Substitution

Note: Most differentiation involves the chain rule, so we should expect that most antidifferentiation will involve

undoing the chain rule [1ex] (the $u$-Substitution)

If $F$ is an antiderivative for $f$, then

$$\frac{d}{dx} [F(u(x))] = F'(u(x)) u'(x) = f(u(x)) u'(x)$$

$$\int f(u(x)) u'(x) dx = \int f(u) du = F(u) + C = F(u(x)) + C.$$ 

The $u$-Substitution
Problem 9

Calculate

1. \[ \int \sin x \cos x \, dx \]
2. \[ \int 2x^3 \sec^2(x^4 + 1) \, dx \]
3. \[ \int \sec^3 x \tan x \, dx \]
4. \[ \int x(x - 3)^5 \, dx \]

Substitution in Definite Integrals

The Change of Variables Formula

\[ \int_a^b f(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du. \]

We change the limits of integration to reflect the substitution.

Problem 10
Evaluate

1. \( \int_0^2 (x^2 - 1)(x^3 - 3x + 2)^3 \, dx \)

2. \( \int_0^{1/2} \cos^3 \pi x \sin \pi x \, dx \)

3. \( \int_0^{\sqrt{3}} x^5 \sqrt{x^2 + 1} \, dx \)

**Definite Integral and Antiderivative**

![](image)

\( F(x) = \) area from \( a \) to \( x \) and \( F(x + h) = \) area from \( a \) to \( x + h \).

Therefore \( F(x + h) - F(x) = \) area from \( x \) to \( x + h \) \( \approx f(x) \, h \) if \( h \) is small and

\[
\frac{F(x + h) - F(x)}{h} = \frac{f(x) \, h}{h} = f(x).
\]

\[
\left. \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) \right|_a^b = f(x).
\]

**Problem 11**

1. Find \( f(x) \) such that \( \int_{-2}^x f(t) \, dt = \cos(2x) + 1 \).

2. Give the function \( f(x) \) that solves the equation \( \int_2^x (t + 1)f(t) \, dt = \sin(x) \).

**Properties**

\[
\frac{d}{dx} \left( \int_a^u f(t) \, dt \right) = f(u) \frac{du}{dx}
\]

\[
\frac{d}{dx} \left( \int_v^b f(t) \, dt \right) = -f(v) \frac{dv}{dx}
\]

\[
\frac{d}{dx} \left( \int_v^u f(t) \, dt \right) = f(u) \frac{du}{dx} - f(v) \frac{dv}{dx}
\]
Problem 12

Find

1. \[ \frac{d}{dx} \left( \int_0^{x^3} \frac{dt}{1 + t} \right) \]

2. \[ \frac{d}{dx} \left( \int_{-3}^{x^2} (3t - \sin(t^2)) \, dt \right) \]

3. \[ \frac{d}{dx} \left( \int_{x}^{2x} \frac{dt}{1 + t^2} \right) \]

Mean-Value Theorems for Integrals

Let \( f_{avg} \) denote the average or mean value of \( f \) on \([a, b]\). Then

\[
f_{avg} = \frac{1}{b - a} \int_a^b f(x) \, dx.
\]

The First Mean-Value Theorems for Integrals

If \( f \) is continuous on \([a, b]\), then there is at least one number \( c \) in \((a, b)\) for which

\[ f(c) = f_{avg}. \]

Problem 13

Give the average value of the function \( f(x) = \sin x \) on the interval \([0, \pi/2]\).

Problem 14

Give the value of \( c \) that satisfies the conclusion of the mean value theorem for integrals for the function \( f(x) = x^2 - 2x + 3 \) on the interval \([1, 4]\).
Area by Integration with Respect to $x$: $f(x) \geq g(x)$

Rectangle Area $[f(x_i^*) - g(x_i^*)]\Delta x_i$

Riemann Sum $\sum[f(x_i^*) - g(x_i^*)]\Delta x_i$
Area by Integration with Respect to $x$: \[ \text{area}(\Omega) = \int_a^b [f(x) - g(x)] \, dx = \lim_{||P|| \to 0} \sum [f(x_i^*) - g(x_i^*)] \Delta x_i. \]

**Area by Integration with Respect to $y$:** $F(y) \geq G(y)$

Rectangle Area $[F(y_i^*) - G(y_i^*)] \Delta y_i$
Riemann Sum: \[ \sum (F(y_i^*) - G(y_i^*)) \Delta y_i \]
area(Ω) = \int_{c}^{d} [F(y) - G(y)] dy = \lim_{\|P\| \to 0} \sum [F(y_i^*) - G(y_i^*)] \Delta y_i.

Problem 15
Give a formula involving integral(s) in x for the region bounded by y = x - 2 and y = \sqrt{x}.
Problem 16
Give a formula involving integral(s) in $y$ for the region bounded by $y = x - 2$ and $y = \sqrt{x}$.

Solid of Revolution About the $x$-Axis: Washer

Cylinder Volume: $\pi([f(x_i^*)]^2 - [g(x_i^*)]^2)\Delta x_i$ [1ex] Riemann Sum:
\[
\sum \pi([f(x_i^*)]^2 - [g(x_i^*)]^2)\Delta x_i
\] [1ex]
\[ V = \int_{a}^{b} \pi ([f(x)]^2 - [g(x)]^2) \, dx = \lim_{\|P\| \to 0} \sum \pi ([f(x_i^*)]^2 - [g(x_i^*)]^2) \Delta x_i \]

**Solid of Revolution About the x-Axis: Shell**

\[ V = \int_{c}^{d} 2\pi y [F(y) - G(y)] \, dy = \lim_{\|P\| \to 0} \sum 2\pi y_i^* [F(y_i^*) - G(y_i^*)] \Delta y_i. \]

The integrand \(2\pi y [F(y) - G(y)]\) is the lateral area of the cylinder.

**Problem 17**

The region bounded by \(y = x^2\) and \(y = 2x\) is rotated around the x-axis. Give a formula involving integrals in \(x\) for the volume of the solid that is generated.
Problem 18
The region bounded by \( y = x^2 \) and \( y = 2x \) is rotated around the \( x \)-axis. Give a formula involving integrals in \( y \) for the volume of the solid that is generated.

Solid of Revolution About the \( y \)-Axis: Washer

Cylinder Volume: \( \pi \left( [F(y_i^*)]^2 - [G(y_i^*)]^2 \right) \Delta y_i \) [1ex] Riemann Sum:
\[
\sum \pi \left( [F(y_i^*)]^2 - [G(y_i^*)]^2 \right) \Delta y_i \] [1ex]
\[ V = \int_c^d \pi (\| F(y) \|^2 - \| G(y) \|^2) \, dy = \lim_{\| P \| \to 0} \sum \pi (\| F(y_i^*) \|^2 - \| G(y_i^*) \|^2) \Delta y_i \]

**Solid of Revolution About the y-Axis: Shell**

\[ V = \int_a^b 2\pi x [f(x) - g(x)] \, dx = \lim_{\| P \| \to 0} \sum 2\pi x_i^* [f(x_i^*) - g(x_i^*)] \Delta x_i. \]

The integrand \( 2\pi x [f(x) - g(x)] \) is the lateral area of the cylinder.

**Problem 19**

The region bounded by \( y = x^2 \) and \( y = 2x \) is rotated around the y-axis. Give a formula involving integrals in y for the volume of the solid that is generated.
Problem 20
The region bounded by \( y = x^2 \) and \( y = 2x \) is rotated around the \( y \)-axis. Give a formula involving integrals in \( x \) for the volume of the solid that is generated.