1 One-To-One Functions

1.1 Definition of the One-To-One Functions

What are One-To-One Functions? Geometric Test

Horizontal Line Test

- If some horizontal line intersects the graph of the function more than once, then the function is not one-to-one.
- If no horizontal line intersects the graph of the function more than once, then the function is one-to-one.

What are One-To-One Functions? Algebraic Test

Definition 1. A function \( f \) is said to be one-to-one (or injective) if

\[
\text{if } x_1 = x_2 \implies f(x_1) = f(x_2).
\]

Lemma 2. The function \( f \) is one-to-one if and only if

\[
\forall x_1, \forall x_2, x_1 \neq x_2 \implies f(x_1) \neq f(x_2).
\]
Examples and Counter-Examples

Examples

- \( f(x) = 3x - 5 \) is 1-to-1.
- \( f(x) = x^3 \) is not 1-to-1.
- \( f(x) = \frac{1}{x} \) is 1-to-1.
- \( f(x) = x^n - x, \ n > 0, \) is not 1-to-1.

Proof.
- \( f(x_1) = f(x_2) \implies 3x_1 - 5 = 3x_2 - 5 \implies x_1 = x_2. \) In general, \( f(x) = ax - b, \ a \neq 0, \) is 1-to-1.

- \( f(1) = (1)^2 = 1 = (-1)^2 = f(-1). \) In general, \( f(x) = x^n, \ n \) even, is not 1-to-1.

- \( f(x_1) = f(x_2) \implies x_1^3 = x_2^3 \implies x_1 = x_2. \) In general, \( f(x) = x^n, \ n \) odd, is 1-to-1.

- \( f(0) = 0^n - 0 = 0 = (1)^n - 1 = f(1). \) In general, 1-to-1 of \( f \) and \( g \) does not always imply 1-to-1 of \( f + g. \)

1.2 Properties of One-To-One Functions

Properties

If \( f \) and \( g \) are one-to-one, then \( f \circ g \) is one-to-one.

Proof. \( f \circ g(x_1) = f \circ g(x_2) \implies f(g(x_1)) = f(g(x_2)) \implies g(x_1) = g(x_2) \implies x_1 = x_2. \)

Examples 4.
- \( f(x) = 3x^3 - 5 \) is one-to-one, since \( f = g \circ u \) where \( g(u) = 3u - 5 \) and \( u(x) = x^3 \) are one-to-one.
- \( f(x) = (3x - 5)^3 \) is one-to-one, since \( f = g \circ u \) where \( g(u) = u^3 \) and \( u(x) = 3x - 5 \) are one-to-one.
- \( f(x) = \frac{1}{3x^3 - 5} \) is one-to-one, since \( f = g \circ u \) where \( g(u) = \frac{1}{u} \) and \( u(x) = 3x^3 - 5 \) are one-to-one.
1.3 Increasing/Decreasing Functions and One-To-Oneness

**Definition 5.**
- A function $f$ is (strictly) increasing if
  \[ \forall x_1, \forall x_2, \; x_1 < x_2 \; \text{implies} \; f(x_1) < f(x_2). \]
- A function $f$ is (strictly) decreasing if
  \[ \forall x_1, \forall x_2, \; x_1 < x_2 \; \text{implies} \; f(x_1) > f(x_2). \]

**Theorem 6.** Functions that are increasing or decreasing are one-to-one.

**Proof.** For $x_1 \neq x_2$, either $x_1 < x_2$ or $x_1 > x_2$ ans so, by monotonicity, either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, thus $f(x_1) \neq f(x_2)$.

**Sign of the Derivative Test for One-To-Oneness**

**Theorem 7.**
- If $f'(x) > 0$ for all $x$, then $f$ is increasing, thus one-to-one.
- If $f'(x) < 0$ for all $x$, then $f$ is decreasing, thus one-to-one.

**Examples 8.**
- $f(x) = x^3 + \frac{1}{2}x$ is one-to-one, since $f'(x) = 3x^2 + \frac{1}{2} > 0$ for all $x$.
- $f(x) = -x^3 - 2x^2 - 2x$ is one-to-one, since $f'(x) = -5x^4 - 6x^2 - 2 < 0$ for all $x$.
- $f(x) = x - \pi + \cos x$ is one-to-one, since $f'(x) = 1 - \sin x \geq 0$ and $f'(x) = 0$ only at $x = \frac{\pi}{2} + 2k\pi$.

2 Inverse Functions

**2.1 Definition of Inverse Functions**

What are Inverse Functions?
Definition 9. Let $f$ be a one-to-one function. The inverse of $f$, denoted by $f^{-1}$, is the unique function with domain equal to the range of $f$ that satisfies

$$f(f^{-1}(x)) = x$$

for all $x$ in the range of $f$.

Warning
DON’T Confuse $f^{-1}$ with the reciprocal of $f$, that is, with $1/f$. The “$-1$” in the notation for the inverse of $f$ is not an exponent; $f^{-1}(x)$ does not mean $1/f(x)$.

Example
Example 10. \( f(x) = x^3 \Rightarrow f^{-1}(x) = x^{1/3}. \)
Proof. • By definition, \( f^{-1} \) satisfies the equation
\[
    f(f^{-1}(x)) = x \quad \text{for all } x.
\]

• Set \( y = f^{-1}(x) \) and solve \( f(y) = x \) for \( y \):
\[
    f(y) = x \quad \Rightarrow \quad y^3 = x \quad \Rightarrow \quad y = x^{1/3}.
\]

• Substitute \( f^{-1}(x) \) back in for \( y \),
\[
    f^{-1}(x) = x^{1/3}. \quad \Box
\]

In general,
\[
    f(x) = x^n, \text{ } n \text{ odd,} \quad \Rightarrow \quad f^{-1}(x) = x^{1/n}.
\]

Example
Example 11. \( f(x) = 3x - 5 \Rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{5}{3} \).

Proof. \( f^{-1} \) by definition, \( f^{-1} \) satisfies \( f(f^{-1}(x)) = x \), \( \forall x \).

- Set \( y = f^{-1}(x) \) and solve \( f(y) = x \) for \( y \):
  
  \[
  f(y) = x \Rightarrow 3y - 5 = x \Rightarrow y = \frac{1}{3}x + \frac{5}{3}.
  \]

- Substitute \( f^{-1}(x) \) back in for \( y \),
  
  \[
  f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}.
  \]


In general,

\[
 f(x) = ax + b, \ a \neq 0, \Rightarrow f^{-1}(x) = \frac{1}{a}x - \frac{b}{a}.
\]

2.2 Properties of Inverse Functions

Undone Properties

\[
 f \circ f^{-1} = \text{Id}_{\mathcal{R}(f)}
\]

\[
 \mathcal{D}(f^{-1}) = \mathcal{R}(f)
\]
Theorem 12. By definition, \( f^{-1} \) satisfies
\[
f(f^{-1}(x)) = x \quad \text{for all } x \text{ in the range of } f.
\]
It is also true that
\[
f^{-1}(f(x)) = x \quad \text{for all } x \text{ in the domain of } f.
\]
Proof. \( \forall x \in \mathcal{D}(f), \) set \( y = f(x). \) Since \( y \in \mathcal{R}(f), \)
\[
f(f^{-1}(y)) = y \quad \Rightarrow \quad f(f^{-1}(f(x))) = f(x).
\]
• \( f \) being one-to-one implies \( f^{-1}(f(x)) = x. \) \( \Box \)

Graphs of \( f \) and \( f^{-1} \)
**Graphs of $f$ and $f^{-1}$**

The graph of $f^{-1}$ is the graph of $f$ reflected in the line $y = x$.

*Example 13.* Given the graph of $f$, sketch the graph of $f^{-1}$.

**Solution**

First draw the line $y = x$. Then reflect the graph of $f$ in that line.

**Corollary 14.** $f$ is continuous $\Rightarrow$ so is $f^{-1}$.

### 2.3 Differentiability of Inverses

**Differentiability of Inverses**

*Theorem 15.*

\[
(f^{-1})'(y) = \frac{1}{f'(x)}, \quad f'(x) \neq 0, \quad y = f(x).
\]

*Proof.* $\forall y \in \mathcal{D}(f^{-1}) = \mathcal{R}(f), \exists x \in \mathcal{D}(f)$ s.t. $y = f(x)$. By definition,

\[
f^{-1}(f(x)) = x \quad \Rightarrow \quad \frac{d}{dx} f^{-1}(f(x)) = (f^{-1})'(f(x))f'(x) = 1.
\]
• If \( f'(x) \neq 0 \), then
\[
(f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad \Rightarrow \quad (f^{-1})'(y) = \frac{1}{f'(x)}.
\]
\[
\square
\]

Example

Example 16. Let \( f(x) = x^3 + \frac{1}{2}x \). Calculate \((f^{-1})'(9)\).

Solution

• Note that \( f'(x) = 3x^2 + \frac{1}{2} > 0 \), thus \( f \) is one-to-one.
• Note that \((f^{-1})'(y) = \frac{1}{f'(x)}, \ y = f(x)\).
• To calculate \((f^{-1})'(y)\) at \( y = 9 \), find a number \( x \) s.t. \( f(x) = 9 \):
\[
f(x) = 9 \quad \Rightarrow \quad x^3 + \frac{1}{2}x = 9 \quad \Rightarrow \quad x = 2.
\]
• Since \( f'(2) = 3(2)^2 + \frac{1}{2} = \frac{25}{2} \), then \((f^{-1})'(9) = \frac{1}{f'(2)} = \frac{2}{25}\).

Note that to calculate \((f^{-1})'(y)\) at a specific \( y \) using
\[
(f^{-1})'(y) = \frac{1}{f'(x)}, \ f'(x) \neq 0, \ y = f(x),
\]
we only need the value of \( x \) s.t. \( f(x) = y \), not the inverse function \( f^{-1} \), which may not be known explicitly.

Daily Grades

Daily Grades

1. \( f(x) = x, \ f^{-1}(x) =? \) : (a) not exist, (b) \( x \), (c) \( \frac{1}{x} \).
2. \( f(x) = x^3, \ f^{-1}(x) =? \) : (a) not exist, (b) \( x^\frac{1}{3} \), (c) \( \frac{1}{x^3} \).
3. \( f(x) = x^2, \ f^{-1}(x) =? \) : (a) not exist, (b) \( x^\frac{1}{2} \), (c) \( \frac{1}{x^2} \).
4. \( f(x) = 3x - 3, \ (f^{-1})'(1) =? \) : (a) not exist, (b) 3, (c) \( \frac{1}{3} \).

Outline
Contents

1 One-To-One Functions 1
  1.1 Definition ............................................. 1
  1.2 Properties ........................................... 2
  1.3 Monotonicity ........................................ 3

2 Inverses 3
  2.1 Definition ............................................. 3
  2.2 Properties ........................................... 8
  2.3 Differentiability .................................... 11