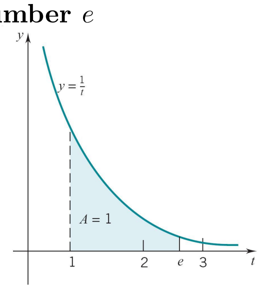
Lecture 4 Section 7.4 The Exponential Function Section 7.5 Arbitrary Powers; Other Bases

Jiwen He

1 Definition and Properties of the Exp Function

1.1 Definition of the Exp Function

Number e



Definition 1. The number e is defined by

$$\ln e = 1$$

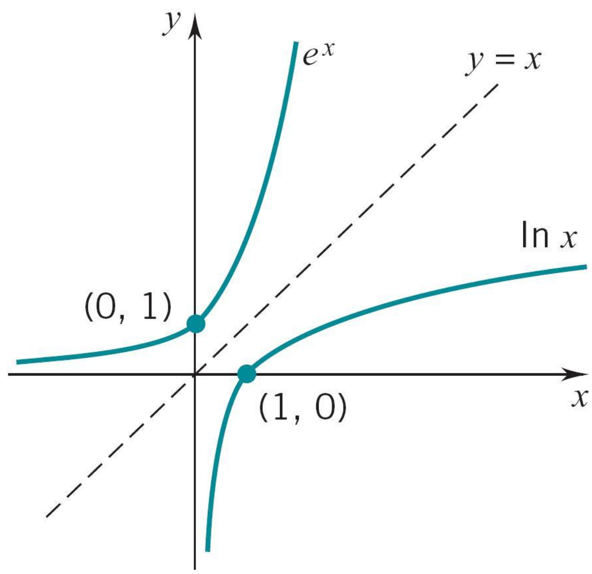
i.e., the unique number at which $\ln x = 1$.

Remark
Let $L(x) = \ln x$ and $E(x) = e^x$ for x rational. Then

$$L \circ E(x) = \ln e^x = x \ln e = x,$$

i.e., $E(x)$ is the inverse of $L(x)$.

e^x: Inverse of $\ln x$



Definition 2. The exp function $E(x) = e^x$ is the *inverse* of the log function $L(x) = \ln x$:

$$L \circ E(x) = \ln e^x = x, \quad \forall x.$$

Properties

- $\ln x$ is the *inverse* of e^x: $\quad \forall x > 0, \quad E \circ L = e^{\ln x} = x$.
- $\forall x > 0$, $y = \ln x \quad \Leftrightarrow \quad e^y = x$.
- graph(e^x) is the reflection of graph($\ln x$) by line $y = x$.
- range(E) = domain(L) = $(0, \infty)$, domain(E) = range(L) = $(-\infty, \infty)$.
- $\lim_{x \to -\infty} e^x = 0 \quad \Leftrightarrow \quad \lim_{x \to 0^+} \ln x = -\infty$, $\lim_{x \to \infty} e^x = \infty \quad \Leftrightarrow \quad \lim_{x \to \infty} \ln x = \infty$.

1.2 Properties of the Exp Function

Algebraic Property

Lemma 3.
- *$e^{x+y} = e^x \cdot e^y$.*
- *$e^{-x} = \dfrac{1}{e^x}$.*
- *$e^{x-y} = \dfrac{e^x}{e^y}$.*
- *$e^{rx} = (e^x)^r$, $\forall r$ rational.*

Proof

$$\ln e^{x+y} = x + y = \ln e^x + \ln e^y = \ln\left(e^x \cdot e^y\right).$$

Since $\ln x$ is one-to-one, then

$$e^{x+y} = e^x \cdot e^y.$$

$$1 = e^0 = e^{x+(-x)} = e^x \cdot e^{-x} \quad \Rightarrow \quad e^{-x} = \frac{1}{e^x}.$$

$$e^{x-y} = e^{x+(-y)} = e^x \cdot e^{-y} = e^x \cdot \frac{1}{e^y} = \frac{e^x}{e^y}.$$

- For $r = m \in \mathbb{N}$, $e^{mx} = e^{\overbrace{x+\cdots+x}^{m}} = \overbrace{e^x \cdots e^x}^{m} = (e^x)^m$.
- For $r = \frac{1}{n}$, $n \in \mathbb{N}$ and $n \neq 0$, $e^x = e^{\frac{n}{n}x} = \left(e^{\frac{1}{n}x}\right)^n \quad \Rightarrow \quad e^{\frac{1}{n}x} = (e^x)^{\frac{1}{n}}$.
- For r rational, let $r = \frac{m}{n}$, $m, n \in \mathbb{N}$ and $n \neq 0$. Then $e^{rx} = e^{\frac{m}{n}x} = \left(e^{\frac{1}{n}x}\right)^m = \left((e^x)^{\frac{1}{n}}\right)^m = (e^x)^{\frac{m}{n}} = (e^x)^r$.

Derivatives

Lemma 4. • $\dfrac{d}{dx}e^x = e^x \quad \Rightarrow \quad \displaystyle\int e^x\,dx = e^x + C.$

- $\dfrac{d^m}{dx^m}e^x = e^x > 0 \quad \Rightarrow \quad E(x) = e^x$ *is concave up, increasing, and positive.*

Proof

Since $E(x) = e^x$ is the inverse of $L(x) = \ln x$, then with $y = e^x$,

$$\frac{d}{dx}e^x = E'(x) = \frac{1}{L'(y)} = \frac{1}{(\ln y)'} = \frac{1}{\frac{1}{y}} = y = e^x.$$

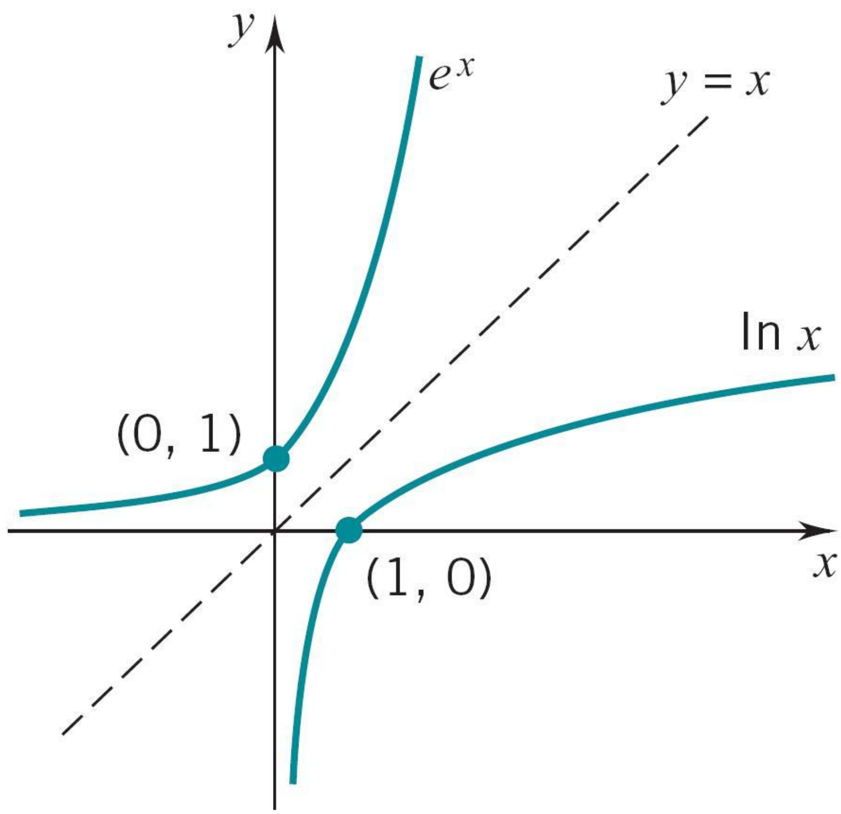
First, for $m = 1$, it is true. Next, assume that it is true for k, then

$$\frac{d^{k+1}}{dx^{k+1}}e^x = \frac{d}{dx}\left(\frac{d^k}{dx^k}e^x\right) = \frac{d}{dx}\left(e^x\right) = e^x.$$

By the axiom of induction, it is true for all positive integer m.

1.3 Another Definition of the Exp Function

e^x: as the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$



Definition 5. (Section 11.5)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{x^k}{k!} \right), \quad \forall x \in R.$$

$(k! = 1 \cdot 2 \cdots k)$

Number e

- $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \right).$
- $e \approx 2.71828182845904523536\ldots$

Limit: $\lim_{x\to\infty}\frac{e^x}{x^n}$

Theorem 6.

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\infty,\quad \forall n\in\mathbb{N}.$$

Proof.

- Recall that

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\cdots.$$

- For large $x>0$,

$$e^x>\frac{x^p}{p!}\quad\Rightarrow\quad\frac{e^x}{x^n}>\frac{x^{p-n}}{p!}.$$

- For $p>n$, $\lim_{x\to\infty}x^{p-n}=\infty$, then $\lim_{x\to\infty}\frac{e^x}{x^n}=\infty$. □

Quiz

Quiz

1. domain of $\ln\left(1+x^2\right)$: (a) $x>1$, (b) $x>-1$, (c) any x.
2. domain of $\ln\left(x\sqrt{4+x^2}\right)$: (a) $x\neq 0$, (b) $x>0$, (c) any x.

2 Differentiation and Graphing

2.1 Chain Rule

Differentiation: Chain Rule

Lemma 7. $\frac{d}{dx}e^u=e^u\frac{du}{dx}$.

Proof

By the chain rule,

$$\frac{d}{dx}e^u=\frac{d}{du}\left(e^u\right)\frac{du}{dx}=e^u\frac{du}{dx}$$

Examples 8.

- $\frac{d}{dx}e^{kx}=e^{kx}\cdot k=ke^{kx}$.
- $\frac{d}{dx}e^{\sqrt{x}}=e^{\sqrt{x}}\cdot\frac{d}{dx}\sqrt{x}=e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}y=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$
- $\frac{d}{dx}e^{-x^2}=e^{-x^2}\frac{d}{dx}\left(-x^2\right)=e^{-x^2}\left(-2x\right)=-2xe^{-x^2}$.

Examples: Chain Rule

Examples 9.

- $\frac{d}{dx}e^{4\ln x}$.
- $\frac{d}{dx}e^{\sin 2x}$.
- $\frac{d}{dx}\ln\left(\cos e^{2x}\right)$.

Solution

Simplify it before the differentiation:

$$e^{4\ln x} = \left(e^{\ln x}\right)^4 = x^4 \quad \Rightarrow \quad \frac{d}{dx}e^{4\ln x} = \frac{d}{dx}x^4 = 4x^3.$$

By the chain rule,

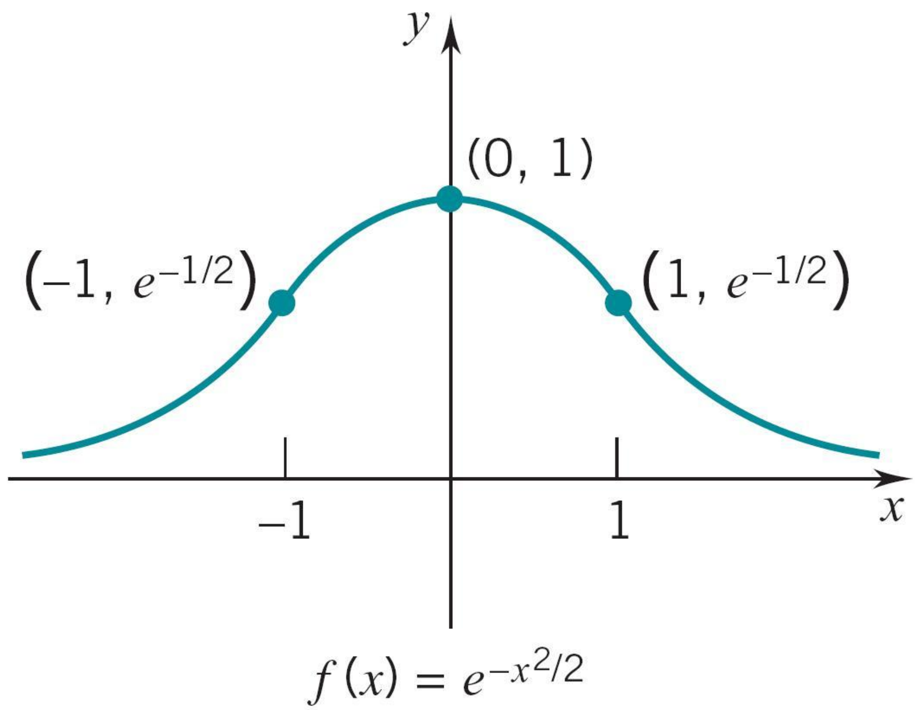
$$\frac{d}{dx}e^{\sin 2x} = e^{\sin 2x}\frac{d}{dx}\sin 2x = e^{\sin 2x}\cdot 2\cos 2x$$

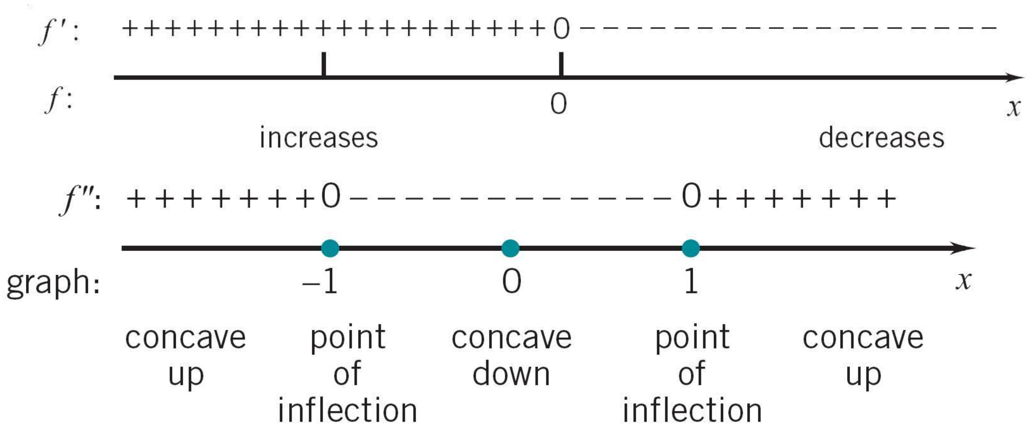
By the chain rule,

$$\frac{d}{dx}\ln\left(\cos e^{2x}\right) = \frac{1}{\cos e^{2x}}\cdot\left(-\sin e^{2x}\right)\cdot\frac{d}{dx}e^{2x} = -2e^{2x}\tan e^{2x}.$$

2.2 Graphing

Graph of $f(x) = e^{-\frac{x^2}{2}}$





Example 10. Let $f(x) = e^{-\frac{x^2}{2}}$. Determine the symmetry of graph and asymptotes. On what intervals does f increase? Decrease? Find the extrem values of f.Determine the concavity and inflection points.

Solution

Since $f(-x) = e^{-\frac{(-x)^2}{2}} = e^{-\frac{x^2}{2}} = f(x)$ and $\lim_{x\to\pm\infty} e^{-\frac{(-x)^2}{2}} = 0$, the graph is symmetry w.r.t. the y-axis, and the x-axis is a horizontal asymptote.

- We have $$f'(x) = e^{-\frac{x^2}{2}}(-x) = -xe^{-\frac{x^2}{2}}.$$
- Thus $f \uparrow$ on $(-\infty, 0)$ and $\downarrow$ on $(0, \infty)$.
- At $x = 0$, $f'(x) = 0$. Thus $$f(0) = e^0 = 1$$ is the (only) local and absolute maximum.
- From $f'(x) = -xe^{-\frac{x^2}{2}}$, we have $$f''(x) = -e^{-\frac{x^2}{2}} + x^2e^{-\frac{x^2}{2}} = (x^2 - 1)e^{-\frac{x^2}{2}}.$$
- At $x = \pm 1$, $f''(x) = 0$. Then, the graph is concave up on $(-\infty, -1)$ and $(1, \infty)$; the graph is concave down on $(-1, 1)$.
- The points $$(\pm 1, f(\pm 1)) = (\pm 1, e^{-\frac{1}{2}})$$ are points of inflection.

Quiz (cont.)

Quiz (cont.)

3. $\frac{d}{dx}(\ln|x|) = ?$: (a) $\frac{1}{x}$, (b) $\frac{1}{|x|}$, (c) $-\frac{1}{x}$.

4. $\int x^{-1}\,dx = ?$: (a) $\ln x + C$, (b) $\ln|x| + C$, (c) $x^{-1} + C$.

3 Integration

3.1 u-Substitution

Integration: u-Substitution
Theorem 11.
$$\int e^{g(x)}g'(x)\,dx = e^{g(x)} + C.$$

Proof.
Let $u = g(x)$, thus $du = g'(x)dx$, then
$$\int e^{g(x)}g'(x)\,dx = \int e^u du = e^u + C = e^{g(x)} + C.$$

Example 12. Calculate $\int xe^{-\frac{x^2}{2}}\,dx$. Let $u = -\frac{x^2}{2}$, thus $du = -xdx$, then
$$\int xe^{-\frac{x^2}{2}}\,dx = -\int e^u du = -e^u + C = -e^{-\frac{x^2}{2}} + C.$$

4 Arbitrary Powers

4.1 Arbitrary Powers

Arbitrary Powers: $f(x) = x^r$
Definition 13. For z irrational, we define $x^z = e^{z \ln x}, \quad x > 0$.

Properties (r and s real numbers)

- For $x > 0$, $x^r = e^{r \ln x}$.
- $x^{r+s} = x^r \cdot x^s, \quad x^{r-s} = \dfrac{x^r}{x^s}, \quad x^{rs} = (x^r)^s$
- $\dfrac{d}{dx} x^r = r x^{r-1}, \quad \Rightarrow \quad \displaystyle\int x^r \, dx = \frac{x^{r+1}}{r+1} + C, \text{ for } r \neq -1.$

Example 14. $\dfrac{d}{dx}\left(x^2+1\right)^{3x} = \dfrac{d}{dx} e^{3x \ln(x^2+1)} = e^{3x \ln(x^2+1)} \dfrac{d}{dx}\left(3x \ln(x^2+1)\right)$
$= e^{3x \ln(x^2+1)} \left(\dfrac{6x^2}{x^2+1} + 3 \ln(x^2+1) \right)$

4.2 Other Bases

Other Bases: $f(x) = p^x$**,** $p > 0$

Definition 15. For $p > 0$, the function

$$f(x) = p^x = e^{x \ln p}$$

is called the exp function with base p.

Properties

$$\frac{d}{dx} p^x = p^x \ln p \quad \Rightarrow \quad \int p^x \, dx = \frac{1}{\ln p} p^x + C, \text{ for } p > 0, \, p \neq 1$$

Other Bases: $f(x) = \log_p x$**,** $p > 0$

Definition 16. For $p > 0$, the function

$$f(x) = \log_p x = \frac{\ln x}{\ln p}$$

is called the log function with base p.

Properties

$$\frac{d}{dx} \log_p x = \frac{1}{x \ln p}.$$

Outline

Contents