

Lecture 4

Section 7.4 The Exponential Function Section 7.5

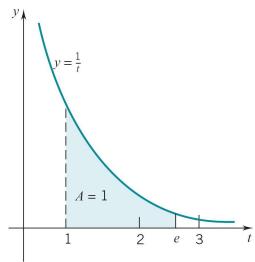
Arbitrary Powers; Other Bases

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1 Definition and Properties of the Exp Function

1.1 Definition of the Exp Function

Number e



Definition 1. The number e is defined by

$$\ln e = 1$$

i.e., the unique number at which $\ln x = 1$.

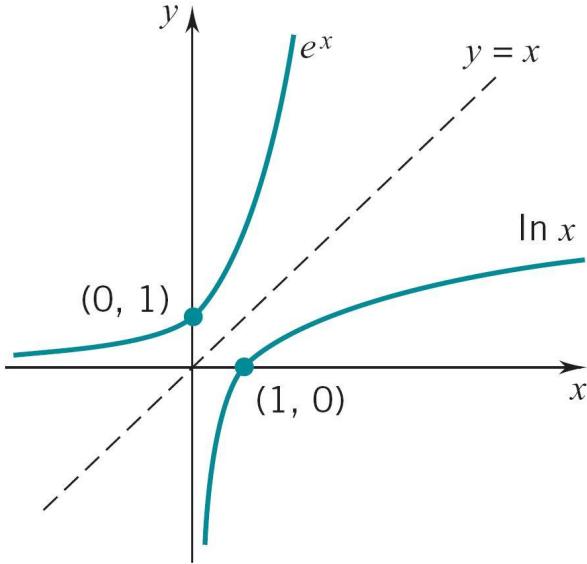
Remark

Let $L(x) = \ln x$ and $E(x) = e^x$ for x rational. Then

$$L \circ E(x) = \ln e^x = x \ln e = x,$$

i.e., $E(x)$ is the inverse of $L(x)$.

e^x : Inverse of $\ln x$



Definition 2. The exp function $E(x) = e^x$ is the *inverse* of the log function $L(x) = \ln x$:

$$L \circ E(x) = \ln e^x = x, \quad \forall x.$$

Properties

- $\ln x$ is the *inverse* of e^x : $\forall x > 0, E \circ L = e^{\ln x} = x$.
- $\forall x > 0, y = \ln x \Leftrightarrow e^y = x$.
- $\text{graph}(e^x)$ is the reflection of $\text{graph}(\ln x)$ by line $y = x$.
- $\text{range}(E) = \text{domain}(L) = (0, \infty)$, $\text{domain}(E) = \text{range}(L) = (-\infty, \infty)$.
- $\lim_{x \rightarrow -\infty} e^x = 0 \Leftrightarrow \lim_{x \rightarrow 0^+} \ln x = -\infty, \lim_{x \rightarrow \infty} e^x = \infty \Leftrightarrow \lim_{x \rightarrow \infty} \ln x = \infty$.

1.2 Properties of the Exp Function

Algebraic Property

Lemma 3. • $e^{x+y} = e^x \cdot e^y$.

- $e^{-x} = \frac{1}{e^x}$.
- $e^{x-y} = \frac{e^x}{e^y}$.
- $e^{rx} = (e^x)^r, \forall r \text{ rational}$.

Proof

$$\ln e^{x+y} = x + y = \ln e^x + \ln e^y = \ln(e^x \cdot e^y).$$

Since $\ln x$ is one-to-one, then

$$e^{x+y} = e^x \cdot e^y.$$

$$1 = e^0 = e^{x+(-x)} = e^x \cdot e^{-x} \Rightarrow e^{-x} = \frac{1}{e^x}.$$

$$e^{x-y} = e^{x+(-y)} = e^x \cdot e^{-y} = e^x \cdot \frac{1}{e^y} = \frac{e^x}{e^y}.$$

- For $r = m \in \mathbb{N}$, $e^{mx} = \overbrace{e^x + \cdots + e^x}^m = \overbrace{e^x \cdots e^x}^m = (e^x)^m$.
- For $r = \frac{1}{n}$, $n \in \mathbb{N}$ and $n \neq 0$, $e^x = e^{\frac{n}{n}x} = \left(e^{\frac{1}{n}x}\right)^n \Rightarrow e^{\frac{1}{n}x} = (e^x)^{\frac{1}{n}}$.
- For r rational, let $r = \frac{m}{n}$, $m, n \in \mathbb{N}$ and $n \neq 0$. Then $e^{rx} = e^{\frac{m}{n}x} = \left(e^{\frac{1}{n}x}\right)^m = \left((e^x)^{\frac{1}{n}}\right)^m = (e^x)^{\frac{m}{n}} = (e^x)^r$.

Derivatives

Lemma 4. • $\frac{d}{dx} e^x = e^x \Rightarrow \int e^x dx = e^x + C$.

- $\frac{d^m}{dx^m} e^x = e^x > 0 \Rightarrow E(x) = e^x$ is concave up, increasing, and positive.

Proof

Since $E(x) = e^x$ is the inverse of $L(x) = \ln x$, then with $y = e^x$,

$$\frac{d}{dx} e^x = E'(x) = \frac{1}{L'(y)} = \frac{1}{(\ln y)'} = \frac{1}{\frac{1}{y}} = y = e^x.$$

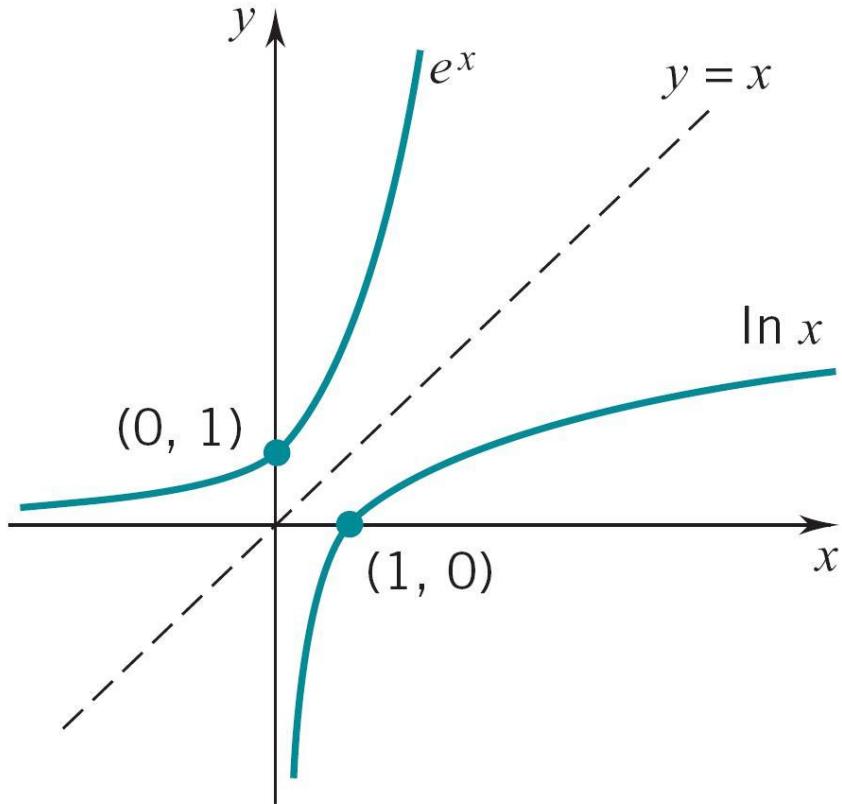
First, for $m = 1$, it is true. Next, assume that it is true for k , then

$$\frac{d^{k+1}}{dx^{k+1}} e^x = \frac{d}{dx} \left(\frac{d^k}{dx^k} e^x \right) = \frac{d}{dx} (e^x) = e^x.$$

By the axiom of induction, it is true for all positive integer m .

1.3 Another Definition of the Exp Function

e^x : as the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$



Definition 5. (Section 11.5)

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \frac{x^k}{k!} \right), \quad \forall x \in R. \end{aligned}$$

$$(k! = 1 \cdot 2 \cdots k)$$

Number e

- $e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots = \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \frac{1}{k!} \right).$

- $e \approx 2.71828182845904523536\dots$

Limit: $\lim_{x \rightarrow \infty} \frac{e^x}{x^n}$

Theorem 6.

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \infty, \quad \forall n \in \mathbb{N}.$$

Proof. • Recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

- For large $x > 0$,

$$e^x > \frac{x^p}{p!} \Rightarrow \frac{e^x}{x^n} > \frac{x^{p-n}}{p!}.$$

- For $p > n$, $\lim_{x \rightarrow \infty} x^{p-n} = \infty$, then $\lim_{x \rightarrow \infty} \frac{e^x}{x^n} = \infty$. \square

Quiz

Quiz

1. domain of $\ln(1+x^2)$: (a) $x > 1$, (b) $x > -1$, (c) any x .
2. domain of $\ln(x\sqrt{4+x^2})$: (a) $x \neq 0$, (b) $x > 0$, (c) any x .

2 Differentiation and Graphing

2.1 Chain Rule

Differentiation: Chain Rule

Lemma 7. $\frac{d}{dx} e^u = e^u \frac{du}{dx}$.

Proof

By the chain rule,

$$\frac{d}{dx} e^u = \frac{d}{du} (e^u) \frac{du}{dx} = e^u \frac{du}{dx}$$

Examples 8. • $\frac{d}{dx} e^{kx} = e^{kx} \cdot k = ke^{kx}$.

$$\bullet \frac{d}{dx} e^{\sqrt{x}} = e^{\sqrt{x}} \cdot \frac{d}{dx} \sqrt{x} = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} y = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$\bullet \frac{d}{dx} e^{-x^2} = e^{-x^2} \frac{d}{dx} (-x^2) = e^{-x^2} (-2x) = -2xe^{-x^2}.$$

Examples: Chain Rule

Examples 9. • $\frac{d}{dx} e^{4 \ln x}$.

- $\frac{d}{dx} e^{\sin 2x}$.

- $\frac{d}{dx} \ln(\cos e^{2x})$.

Solution

Simplify it before the differentiation:

$$e^{4 \ln x} = (e^{\ln x})^4 = x^4 \quad \Rightarrow \quad \frac{d}{dx} e^{4 \ln x} = \frac{d}{dx} x^4 = 4x^3.$$

By the chain rule,

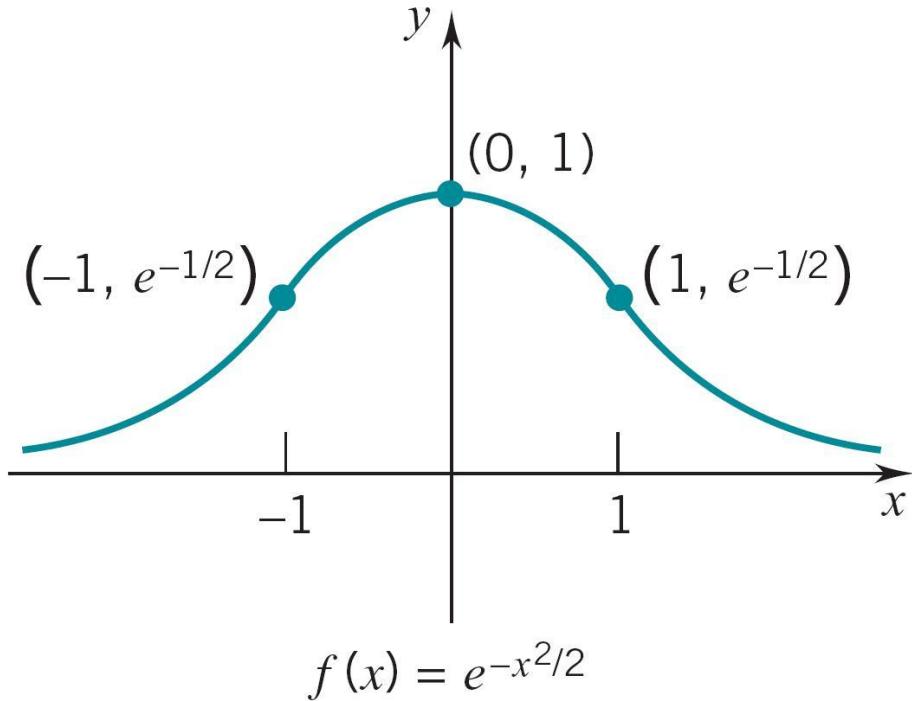
$$\frac{d}{dx} e^{\sin 2x} = e^{\sin 2x} \frac{d}{dx} \sin 2x = e^{\sin 2x} \cdot 2 \cos 2x$$

By the chain rule,

$$\frac{d}{dx} \ln(\cos e^{2x}) = \frac{1}{\cos e^{2x}} \cdot (-\sin e^{2x}) \cdot \frac{d}{dx} e^{2x} = -2e^{2x} \tan e^{2x}.$$

2.2 Graphing

Graph of $f(x) = e^{-\frac{x^2}{2}}$



The figure shows a graph of a function f plotted against x . The horizontal axis is labeled x at the far right. The graph consists of three parts: a dashed line for f' , a solid line for f , and a dashed line for f'' .

- Derivative f' :** A dashed horizontal line with a tick mark at $x = 0$. The label f' is to the left of the tick mark.
- Function f :** A solid horizontal line starting from the left, passing through the origin $(0,0)$, and extending to the right. The label f is to the left of the origin. The word "increases" is written above the line to the left of the origin, and "decreases" is written below the line to the right of the origin.
- Second derivative f'' :** A dashed horizontal line with tick marks at $x = -1$, $x = 0$, and $x = 1$. The label f'' is to the left of the tick mark at $x = -1$. The region between $x = -1$ and $x = 0$ is labeled "0". The region between $x = 0$ and $x = 1$ is also labeled "0". The region to the right of $x = 1$ is labeled "0".
- Graph:** A solid horizontal line with three red dots at $x = -1$, $x = 0$, and $x = 1$. The label "graph:" is to the left of the tick mark at $x = -1$. Below the graph, five regions are labeled: "concave up" under the interval $x < -1$, "point of inflection" under the tick mark at $x = -1$, "concave down" under the interval $-1 < x < 0$, "point of inflection" under the tick mark at $x = 0$, and "concave up" under the interval $x > 1$.

Example 10. Let $f(x) = e^{-\frac{x^2}{2}}$. Determine the symmetry of graph and asymptotes. On what intervals does f increase? Decrease? Find the extrem values of f . Determine the concavity and inflection points.

Solution

Since $f(-x) = e^{-\frac{(-x)^2}{2}} = e^{-\frac{x^2}{2}} = f(x)$ and $\lim_{x \rightarrow \pm\infty} e^{-\frac{(-x)^2}{2}} = 0$, the graph is symmetry w.r.t. the y -axis, and the x -axis is a horizontal asymptote.

- We have $f'(x) = e^{-\frac{x^2}{2}}(-x) = -xe^{-\frac{x^2}{2}}$.
- Thus $f \uparrow$ on $(-\infty, 0)$ and \downarrow on $(0, \infty)$.
- At $x = 0$, $f'(x) = 0$. Thus $f(0) = e^0 = 1$
is the (only) local and absolute maximum.
- From $f'(x) = -xe^{-\frac{x^2}{2}}$, we have $f''(x) = -e^{-\frac{x^2}{2}} + x^2e^{-\frac{x^2}{2}} = (x^2 - 1)e^{-\frac{x^2}{2}}$.
- At $x = \pm 1$, $f''(x) = 0$. Then, the graph is concave up on $(-\infty, -1)$ and $(1, \infty)$; the graph is concave down on $(-1, 1)$.
- The points $(\pm 1, f(\pm 1)) = (\pm 1, e^{-\frac{1}{2}})$
are points of inflection.

Quiz (cont.)

Quiz (cont.)

3. $\frac{d}{dx}(\ln|x|) = ? :$ (a) $\frac{1}{x}$, (b) $\frac{1}{|x|}$, (c) $-\frac{1}{x}$.
4. $\int x^{-1} dx = ? :$ (a) $\ln x + C$, (b) $\ln|x| + C$, (c) $x^{-1} + C$.

3 Integration

3.1 u -Substitution

Integration: u -Substitution

Theorem 11.

$$\int e^{g(x)} g'(x) dx = e^{g(x)} + C.$$

Proof.

Let $u = g(x)$, thus $du = g'(x)dx$, then

$$\int e^{g(x)} g'(x) dx = \int e^u du = e^u + C = e^{g(x)} + C.$$

Example 12. Calculate $\int xe^{-\frac{x^2}{2}} dx$. Let $u = -\frac{x^2}{2}$, thus $du = -xdx$, then

$$\int xe^{-\frac{x^2}{2}} dx = - \int e^u du = -e^u + C = -e^{-\frac{x^2}{2}} + C.$$

4 Arbitrary Powers

4.1 Arbitrary Powers

Arbitrary Powers: $f(x) = x^r$

Definition 13. For z irrational, we define $x^z = e^{z \ln x}$, $x > 0$.

Properties (r and s real numbers)

- For $x > 0$, $x^r = e^{r \ln x}$.
- $x^{r+s} = x^r \cdot x^s$, $x^{r-s} = \frac{x^r}{x^s}$, $x^{rs} = (x^r)^s$
- $\frac{d}{dx} x^r = r x^{r-1}$, $\Rightarrow \int x^r dx = \frac{x^{r+1}}{r+1} + C$, for $r \neq -1$.

$$\begin{aligned} \text{Example 14. } \frac{d}{dx} (x^2 + 1)^{3x} &= \frac{d}{dx} e^{3x \ln(x^2 + 1)} = e^{3x \ln(x^2 + 1)} \frac{d}{dx} (3x \ln(x^2 + 1)) \\ &= e^{3x \ln(x^2 + 1)} \left(\frac{6x^2}{x^2 + 1} + 3 \ln(x^2 + 1) \right) \end{aligned}$$

4.2 Other Bases

Other Bases: $f(x) = p^x$, $p > 0$

Definition 15. For $p > 0$, the function

$$f(x) = p^x = e^{x \ln p}$$

is called the exp function with base p .

Properties

$$\frac{d}{dx} p^x = p^x \ln p \Rightarrow \int p^x dx = \frac{1}{\ln p} p^x + C, \text{ for } p > 0, p \neq 1$$

Other Bases: $f(x) = \log_p x$, $p > 0$

Definition 16. For $p > 0$, the function

$$f(x) = \log_p x = \frac{\ln x}{\ln p}$$

is called the log function with base p .

Properties

$$\frac{d}{dx} \log_p x = \frac{1}{x \ln p}.$$

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