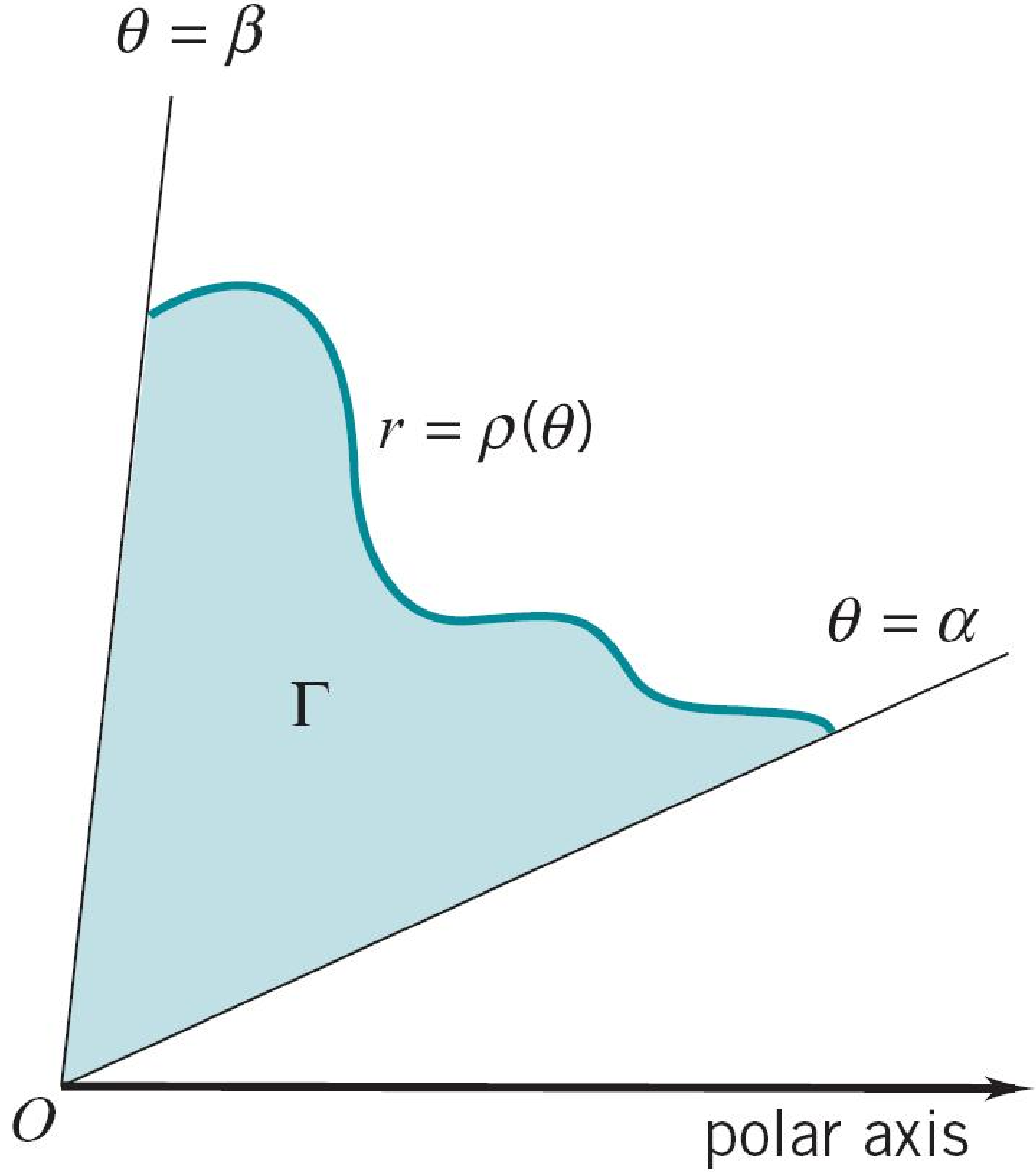
Lecture 13 Section 9.5 Area in Polar Coordinates

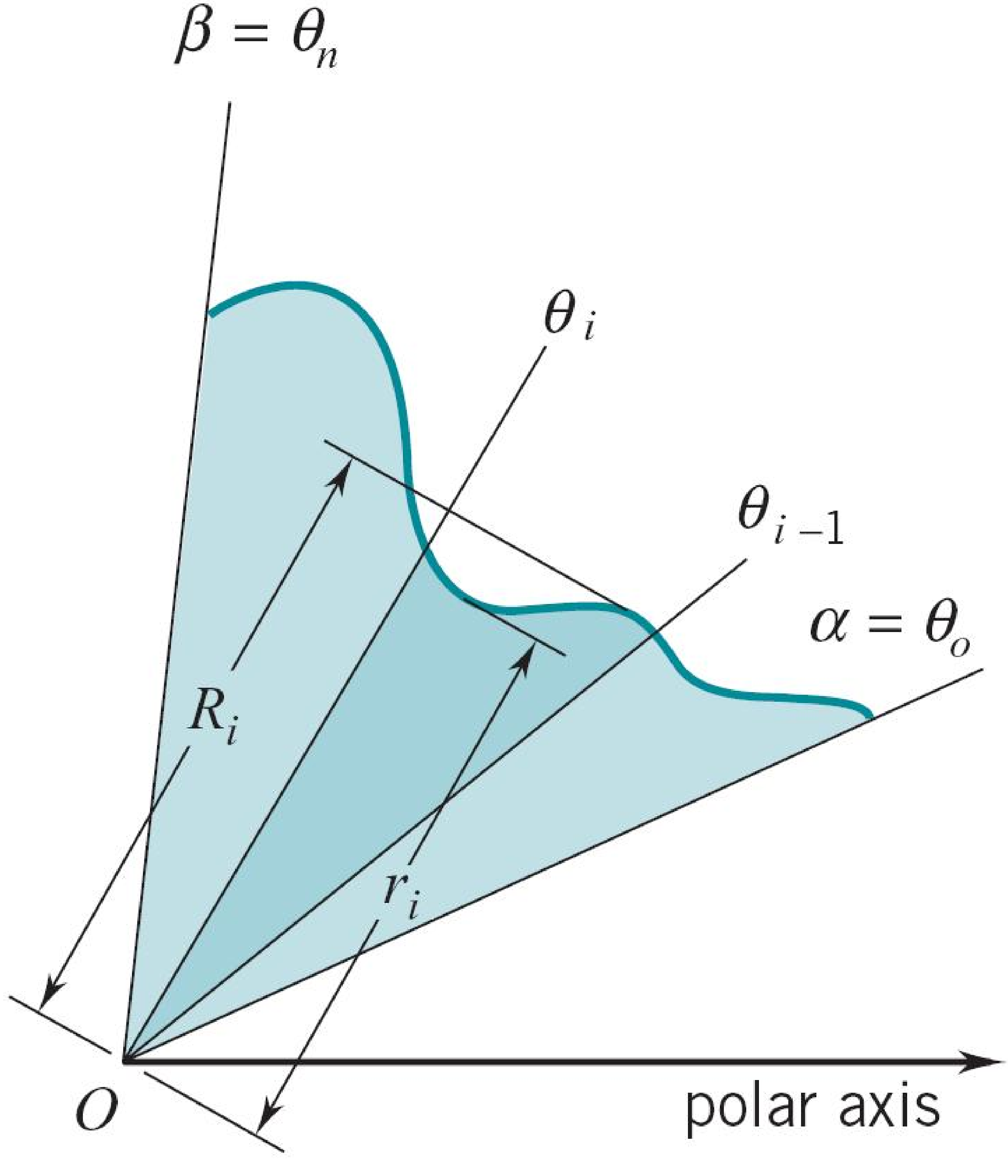
Jiwen He

1 Area of a Polar Region

1.1 Basic Polar Area

Area of a Polar Region


$\theta = \beta$
$r = \rho(\theta)$
$\theta = \alpha$
Γ
O
polar axis



The *area of the polar region* Γ generated by $r = \rho(\theta), \quad \alpha \leq \theta \leq \beta$

is $$A = \int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2\, d\theta$$

Proof

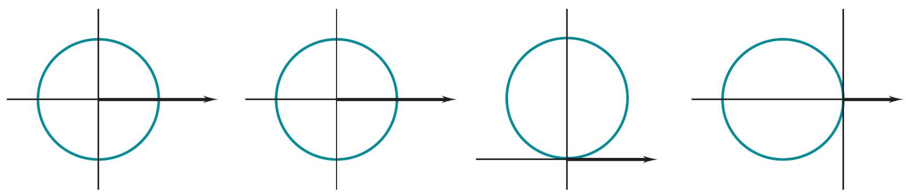
Let $P = \{\theta_0, \theta_1, \cdots, \theta_n\}$ be a partition of $[\alpha, \beta]$. Set $r_i = \min_{\alpha \le \theta \le \beta} \rho(\theta)$ and $R_i = \max_{\alpha \le \theta \le \beta} \rho(\theta)$. Then $$\frac{1}{2} r_i^2 \Delta\theta_i \le A_i \le \frac{1}{2} R_i^2 \Delta\theta_i$$

Summing from $i = 1$ to $i = n$ yields $$L_f(P) \le A \le U_f(P) \quad \text{with} \quad f(\theta) = \frac{1}{2}\big[\rho(\theta)\big]^2$$

Since P is arbitrary, we conclude $$A = \int_{\alpha}^{\beta} f(\theta)\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2\, d\theta$$

1.2 Circles

Area of a Circle of Radius a: $A = \pi a^2$



Circle in Polar Coordinates

$$r = a, \quad 0 \le \theta \le 2\pi$$
$$r = -a, \quad 0 \le \theta \le 2\pi$$
$$r = 2a\sin\theta, \quad 0 \le \theta \le \pi$$
$$r = -2a\cos\theta, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2\, d\theta = \int_0^{2\pi} \frac{1}{2}\big[a\big]^2\, d\theta = \frac{1}{2}a^2 \cdot 2\pi = \pi a^2$$

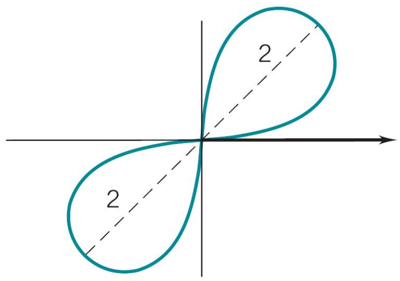
$$A = \int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2\, d\theta = \int_0^{2\pi} \frac{1}{2}\big[-a\big]^2\, d\theta = \frac{1}{2}a^2 \cdot 2\pi = \pi a^2$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2\, d\theta = \int_0^{\pi} \frac{1}{2}\big[2a\sin\theta\big]^2 d\theta = 2a^2\int_0^{\pi} \sin^2\theta d\theta$$
$$= 2a^2\int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = 2a^2\int_0^{\pi} \frac{1}{2}\, d\theta = 2a^2 \cdot \frac{\pi}{2} = \pi a^2$$

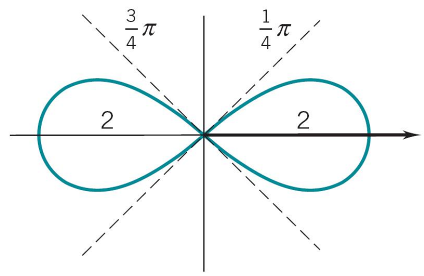
$$A = \int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2\, d\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}\big[-2a\cos\theta\big]^2 d\theta = 2a^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\theta d\theta$$
$$= 2a^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta = 2a^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2}\, d\theta = 2a^2 \cdot \frac{\pi}{2} = \pi a^2$$

1.3 Ribbons

Area of a Lemniscate (Ribbon): $A = a^2$



$r^2 = 4 \sin 2\theta$



$r^2 = 4 \cos 2\theta$

Ribbon
Sketch $r^2 = a^2 \cos 2\theta$ in *4 stages*: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$ Sketch $r^2 = a^2 \sin 2\theta$ in *4 stages*: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$

$$A = 4\int_{\alpha}^{\beta} \frac{1}{2} \left[\rho(\theta)\right]^2 d\theta = 4\int_{0}^{\frac{\pi}{4}} \frac{1}{2} \left(a^2 \cos 2\theta\right) d\theta = 2a^2 \left[\frac{1}{2} \sin 2\theta\right]_0^{\frac{\pi}{4}} = a^2$$

$$A = 4\int_{\alpha}^{\beta} \frac{1}{2} \left[\rho(\theta)\right]^2 d\theta = 4\int_{0}^{\frac{\pi}{4}} \frac{1}{2} \left(a^2 \sin 2\theta\right) d\theta = 2a^2 \left[-\frac{1}{2} \cos 2\theta\right]_0^{\frac{\pi}{4}} = a^2$$

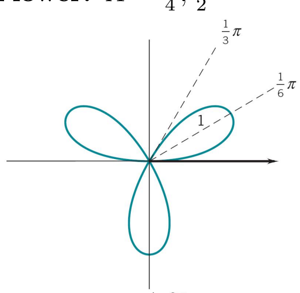
Quiz

Quiz

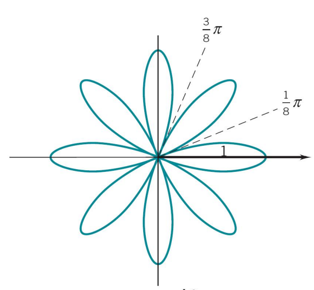
1. $r = 2a \sin\theta$ is a (a) line, (b) circle, (c) lemniscate.
2. $r^2 = a^2 \sin 2\theta$ is a (a) line, (b) circle, (c) lemniscate.

1.4 Flowers

Area of a Flower: $A = \frac{\pi}{4}, \frac{\pi}{2}$



$r = \sin 3\theta$



$r = \cos 4\theta$

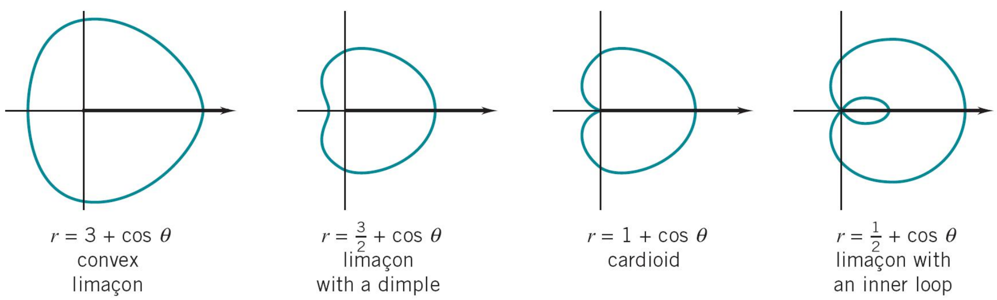
Flower
Sketch $r = \sin 3\theta$ in *6 stages*: $[0, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{3}]$, $\cdots$, $[\frac{2\pi}{3}, \frac{5\pi}{6}]$, $[\frac{5\pi}{6}, \pi]$ Sketch $r = \cos 4\theta$ in *16 stages*: $[0, \frac{\pi}{8}]$, $[\frac{\pi}{8}, \frac{\pi}{4}]$, $\cdots$, $[\frac{15\pi}{8}, 2\pi]$

$$\begin{aligned} A &= 6\int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2 d\theta = 6\int_0^{\frac{\pi}{6}} \frac{1}{2}\big[\sin 3\theta\big]^2 d\theta \\ &= 3\int_0^{\frac{\pi}{6}} \big(\frac{1}{2} - \frac{1}{2}\cos 6\theta\big) d\theta = \frac{3}{2}\Big[\theta - \frac{1}{6}\sin 6\theta\Big]_0^{\frac{\pi}{6}} = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} A &= 16\int_{\alpha}^{\beta} \frac{1}{2}\big[\rho(\theta)\big]^2 d\theta = 16\int_0^{\frac{\pi}{8}} \frac{1}{2}\big[\cos 4\theta\big]^2 d\theta \\ &= 8\int_0^{\frac{\pi}{8}} \big(\frac{1}{2} + \frac{1}{2}\cos 8\theta\big) d\theta = 4\Big[\theta + \frac{1}{8}\sin 8\theta\Big]_0^{\frac{\pi}{8}} = \frac{\pi}{2} \end{aligned}$$

1.5 Limaçons

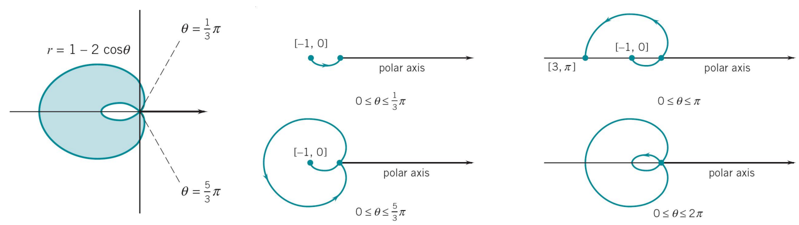
Limaçons (Snails): $r = a + \cos\theta$**,** $a \geq 1$



Flower
Sketch $r = a + \cos\theta$, $a \geq 1$, in *2 stages*: $[0, \pi]$, $[\pi, 2\pi]$

$$\begin{aligned} A &= 2\int_0^{\pi} \frac{1}{2}\big[a + \cos\theta\big]^2 d\theta = \int_0^{\pi} \big(a^2 + 2a\cos\theta + \cos^2\theta\big) d\theta \\ &= \int_0^{\pi} \big(a^2 + 2a\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\big) d\theta = \Big[\big(a^2 + \frac{1}{2}\big)\theta\Big]_0^{\pi} = \big(a^2 + \frac{1}{2}\big)\pi \end{aligned}$$

Limaçon (Snail): $r = 1 - 2\cos\theta$



Area between the Inner and Outer Loops

- Sketch in 4 stages: $[0, \frac{1}{3}\pi]$, $[\frac{1}{3}\pi, \pi]$, $[\pi, \frac{5}{3}\pi]$, $[\frac{5}{3}\pi, 2\pi]$.
- $A = A_{\text{outer}} - A_{\text{inner}} = \left(2\pi + \frac{3}{2}\sqrt{3}\right) - A_{\text{inner}} = \left(2\pi + \frac{3}{2}\sqrt{3}\right) - \left(\pi - \frac{3}{2}\sqrt{3}\right) = \pi + 3\sqrt{3}$

Area Within Outer Loop: A_{outer}

$$A_{\text{outer}} = 2\int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}\left[1 - 2\cos\theta\right]^2 d\theta = \int_{\frac{\pi}{3}}^{\pi} \left(1 - 4\cos\theta + 4\cos^2\theta\right) d\theta$$
$$= \int_{\frac{\pi}{3}}^{\pi} \left(1 - 4\cos\theta + 2 + 2\cos 2\theta\right) d\theta = \Big[3\theta - 4\sin\theta + \sin 2\theta\Big]_{\frac{\pi}{3}}^{\pi} = \cdots$$

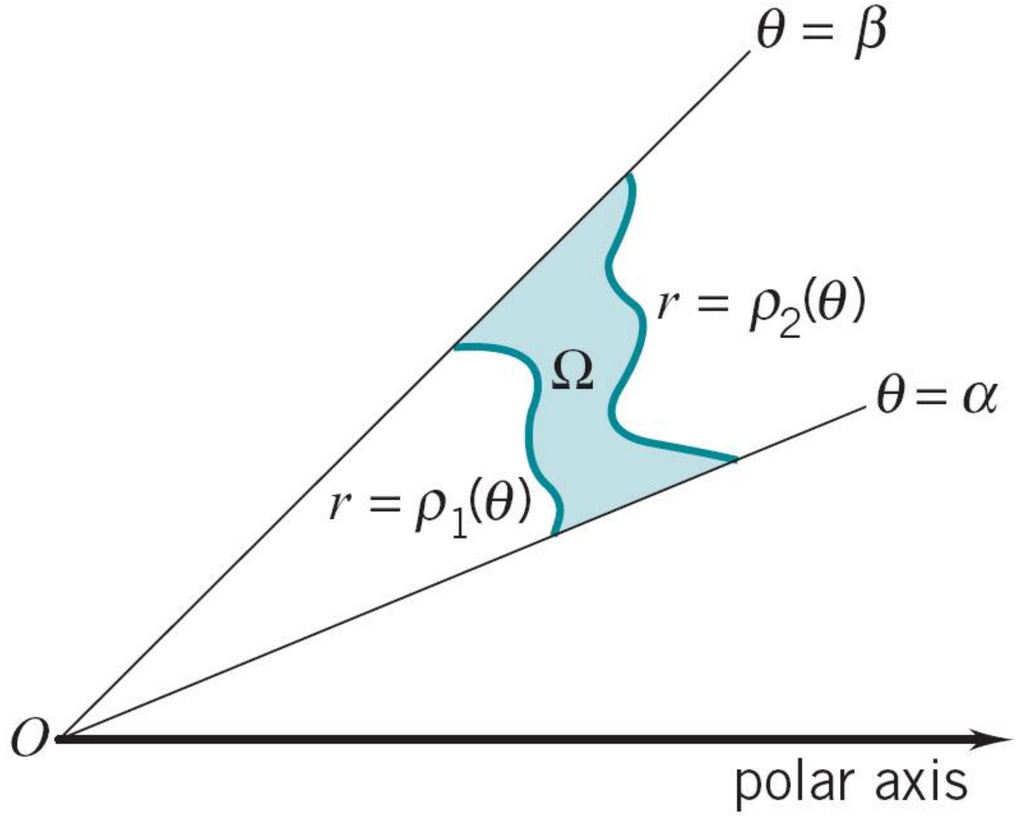
Area Within Inner Loop: A_{inner}

$$A_{\text{inner}} = 2\int_{0}^{\frac{\pi}{3}} \frac{1}{2}\left[1 - 2\cos\theta\right]^2 d\theta = \int_{0}^{\frac{\pi}{3}} \left(1 - 4\cos\theta + 4\cos^2\theta\right) d\theta$$
$$= \int_{0}^{\frac{\pi}{3}} \left(1 - 4\cos\theta + 2 + 2\cos 2\theta\right) d\theta = \Big[3\theta - 4\sin\theta + \sin 2\theta\Big]_{0}^{\frac{\pi}{3}} = \cdots$$

2 Area between Polar Curves

2.1 Between Polar Curves

Area between Polar Curves



Area between $r = \rho_1(\theta)$ **and** $r = \rho_2(\theta)$

$$\text{area of } \Omega = \int_{\alpha_2}^{\beta_2} \frac{1}{2}\big[\rho_2(\theta)\big]^2\, d\theta - \int_{\alpha_1}^{\beta_1} \frac{1}{2}\big[\rho_1(\theta)\big]^2\, d\theta$$

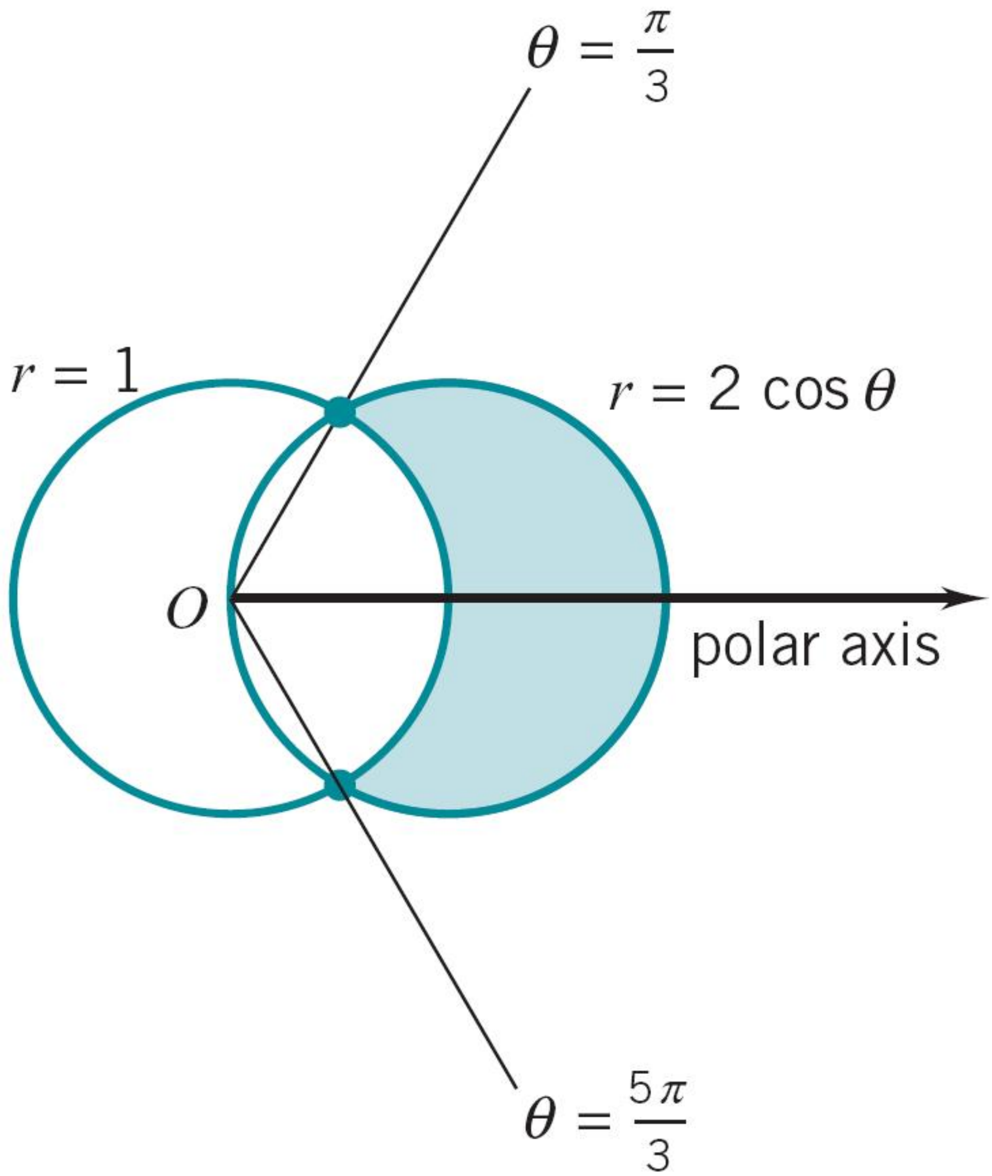
$$= \int_{\alpha}^{\beta} \frac{1}{2}\Big(\big[\rho_2(\theta)\big]^2 - \big[\rho_1(\theta)\big]^2\Big)\, d\theta$$

$$\text{if} \quad \alpha_1 = \alpha_2 = \alpha, \quad \beta_1 = \beta_2 = \beta.$$

Remark
Extra care is needed to determine the intervals of θ values (e.g, $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$) over which the outer and inner boundaries of the region are traced out.

2.2 Between Circles

Area between Circles: $r = 2\cos\theta$ **and** $r = 1$



Area between $r = 2\cos\theta$ **and** $r = 1$

- The two intersection points: $2\cos\theta = 1 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{3}, \frac{5\pi}{3}$ $\Rightarrow \quad [\alpha, \beta] = [-\frac{\pi}{3}, \frac{\pi}{3}]$

- By symmetry,

$$
\begin{aligned}
&\text{area of } \Omega \\
&= \int_{\alpha}^{\beta} \frac{1}{2} \Big(\big[\rho_2(\theta)\big]^2 - \big[\rho_1(\theta)\big]^2 \Big) \, d\theta \\
&= 2 \int_{0}^{\frac{\pi}{3}} \frac{1}{2} \Big(\big[2\cos\theta\big]^2 - \big[1\big]^2 \Big) \, d\theta \\
&= \int_{0}^{\frac{\pi}{3}} \big(1 + 2\cos 2\theta\big) \, d\theta \\
&= \Big[\theta + \sin 2\theta\Big]_0^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}
\end{aligned}
$$

Quiz

Quiz

3. area of $r = 2a\sin\theta$ is :

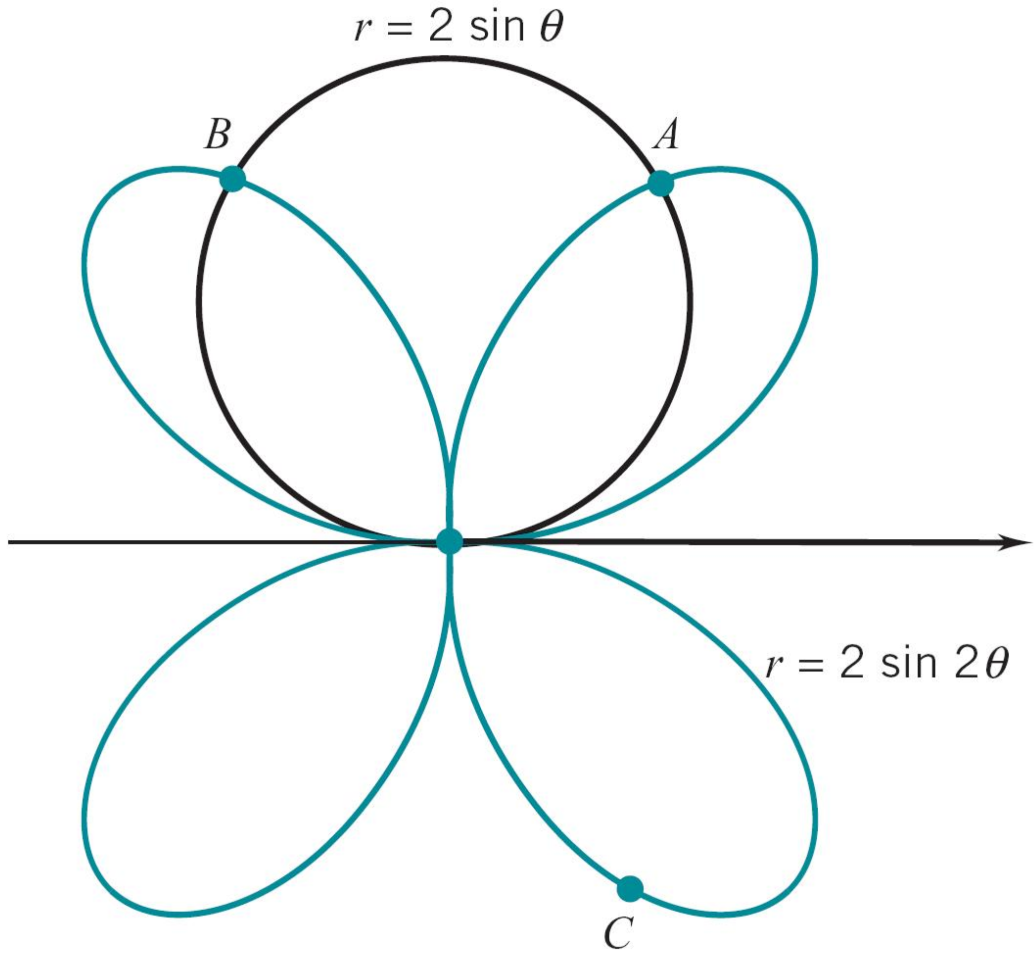
$$\text{(a) } \pi a^2, \quad \text{(b) } \frac{1}{2}\pi a^2, \quad \text{(c) } a^2.$$

4. area of $r^2 = a^2 \sin 2\theta$ is :

$$\text{(a) } \pi a^2, \quad \text{(b) } \frac{1}{2}\pi a^2, \quad \text{(c) } a^2.$$

2.3 Between Circle and Flower

Area outside Circle $r = 2\sin\theta$ **and inside flower** $r = 2\sin 2\theta$

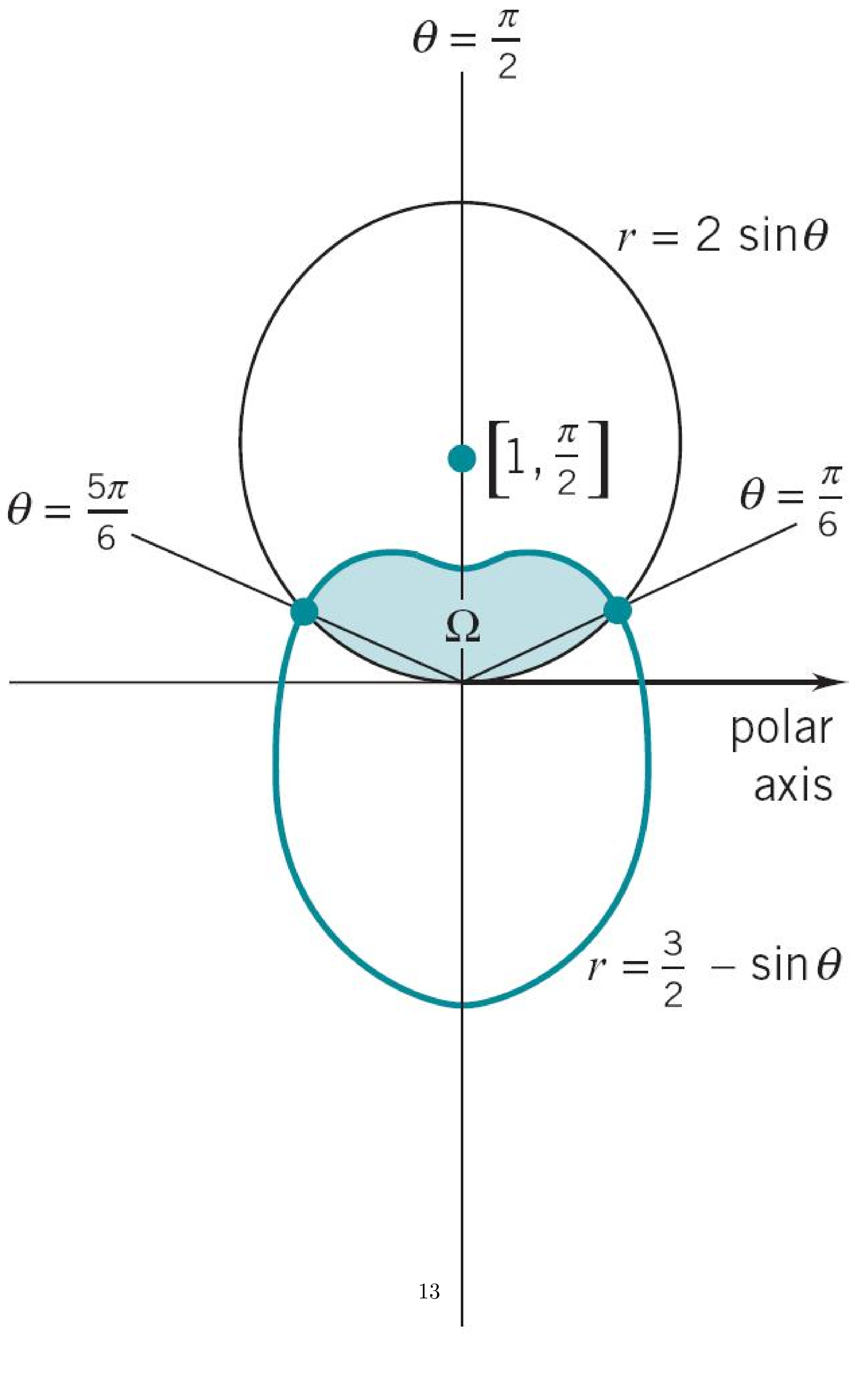


- The three intersection points: $2\sin 2\theta = 2\sin\theta \Rightarrow 2\sin\theta\cos\theta = \sin\theta \Rightarrow \sin\theta\big(2\cos\theta - 1\big) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow 0$, or $\frac{\pi}{3}$, $\frac{5\pi}{3} \Rightarrow [\alpha,\beta] = [0, \frac{\pi}{3}]$ and $[\alpha,\beta] = [\frac{5\pi}{3}, \pi]$
- By symmetry,

$$\begin{aligned} A_1 &= \int_\alpha^\beta \frac{1}{2}\left(\big[\rho_2(\theta)\big]^2 - \big[\rho_1(\theta)\big]^2\right) d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{1}{2}\left(\big[2\sin 2\theta\big]^2 - \big[2\sin\theta\big]^2\right) d\theta \\ &= \int_0^{\frac{\pi}{3}} \big(\cos 2\theta - \cos 4\theta\big)\, d\theta \\ &= \left[\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{3}} = \frac{3\sqrt{3}}{8} \end{aligned}$$

2.4 Between Circle and Limaçon

Area between Circle $r = 2\sin\theta$ and Limaçon $r = \frac{3}{2} - \sin\theta$



$\theta = \frac{\pi}{2}$
$r = 2\ \sin\theta$
$\left[1, \frac{\pi}{2}\right]$
$\theta = \frac{5\pi}{6}$
$\theta = \frac{\pi}{6}$
Ω
polar
axis
$r = \frac{3}{2}\ - \sin\theta$

- The two intersection points: $2\sin\theta = \frac{3}{2} - \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \frac{\pi}{6},\ \frac{5\pi}{6}$
- The area can be represented as follows:

$$\begin{aligned} A &= \int_0^{\frac{\pi}{6}} \frac{1}{2}\big[2\sin\theta\big]^2\,d\theta \\ &+ \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\big[\frac{3}{2} - \sin\theta\big]^2\,d\theta \\ &+ \int_{\frac{5\pi}{6}}^{\pi} \frac{1}{2}\big[2\sin\theta\big]^2\,d\theta \\ &= \cdots = \frac{5}{4}\pi - \frac{15}{8}\sqrt{3} \end{aligned}$$

Outline

Contents