Lecture 14
Section 9.6 Curves Given Parametrically

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A parametrized Curve is a path in the xy-plane traced out by the point \((x(t), y(t))\) as the parameter \(t\) ranges over an interval \(I\).

\[ C = \{(x(t), y(t)) : t \in I\} \]

**Examples**

- The graph of a function \(y = f(x), x \in I\), is a curve \(C\) that is parametrized by
  \[
  x(t) = t, \quad y(t) = f(t), \quad t \in I.
  \]

- The graph of a polar equation \(r = \rho(\theta), \theta \in I\), is a curve \(C\) that is parametrized by the functions
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  x(t) = r \cos t = \rho(t) \cos t, \quad y(t) = r \sin t = \rho(t) \sin t, \quad t \in I.
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Example: Line Segment

We parametrize the line segment in different ways and interpret each parametrization as the motion of a particle with the parameter $t$ being time.
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Examples: Line Segment

Line Segment: $y = 2x, x \in [1, 3]$

- Set $x(t) = t$, then $y(t) = 2t, t \in [1, 3]$
- Set $x(t) = t + 1$, then $y(t) = 2t + 2, t \in [0, 2]$
- Set $x(t) = 3 - t$, then $y(t) = 6 - 2t, t \in [0, 2]$
- Set $x(t) = 3 - 4t$, then $y(t) = 6 - 8t, t \in [0, 1/2]$
- Set $x(t) = 2 - \cos t$, then $y(t) = 4 - 2 \cos t, t \in [0, 4\pi]$

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Example: Parabola

Parabola Arc: \( x = 1 - y^2, \ -1 \leq y \leq 1 \)

- Set \( y(t) = t \), then \( x(t) = 1 - t^2, \ t \in [-1, 1] \) \( \Rightarrow \) changing the domain to all real \( t \) gives us the whole parabola.
- Set \( y(t) = \cos t \), then \( x(t) = 1 - \cos^2 t, \ t \in [0, \pi] \) \( \Rightarrow \) changing the domain to all real \( t \) does not give us any more of the parabola.
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**Example: Spiral of Archimedes**

The curve is a nonending spiral. Here it is shown in detail from $\theta = 0$ to $\theta = 2\pi$.

The parametric representation is

$$x(t) = t \cos t, \quad y(t) = t \sin t, \quad t \geq 0.$$
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Example: Limaçons

Limaçons (Snails): \( r = a + b \cos \theta \)

The parametric representation is

\[
\begin{align*}
x(t) &= (a + b \cos t) \cos t, \\
y(t) &= (a + b \cos t) \sin t,
\end{align*}
\]

\( t \in [0, 2\pi] \).
Example: Petal Curves

Petal Curves (Flowers): $r = a \cos n\theta$, $r = a \sin n\theta$

The parametric representations are

$x(t) = (a \cos(nt)) \cos t, \quad y(t) = (a \cos(nt)) \sin t, \quad t \in [0, 2\pi].$

$x(t) = (a \sin(nt)) \cos t, \quad y(t) = (a \sin(nt)) \sin t, \quad t \in [0, 2\pi].$
Circles: $C = \{ P : d(P, O) = |a| \}$

Center $O$ at $(0,0) \Rightarrow x^2 + y^2 = a^2 \Rightarrow r = a$

$\Rightarrow t \in [0, 2\pi], \quad x(t) = a \cos t, \quad y(t) = a \sin t$

Center $O$ at $(0,a) \Rightarrow x^2 + (y - a)^2 = a^2 \Rightarrow r = 2a \sin \theta$

$\Rightarrow t \in [0, \pi], \quad \begin{cases} x(t) = 2a \sin t \cos t = a \sin 2t, \\ y(t) = 2a \sin t \sin t = a(1 - \cos 2t). \end{cases}$

Another parametric representation is by translation

$\Rightarrow t \in [0, 2\pi], \quad x(t) = a \cos t, \quad y(t) = a \sin t + a$
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Center $O$ at $(a, 0) \Rightarrow (x - a)^2 + y^2 = a^2 \Rightarrow r = 2a \cos \theta$

$\Rightarrow t \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right], \quad \left\{ \begin{array}{l} x(t) = 2a \cos t \cos t = a(1 + \cos 2t), \\ y(t) = 2a \cos t \sin t = a \sin 2t. \end{array} \right.$

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$\Rightarrow t \in [0, 2\pi], \quad x(t) = a \cos t + a, \quad y(t) = a \sin t.$
A **ellipse** is the set of points $P$ in a plane that the sum of whose distances from two fixed points (the foci $F_1$ and $F_2$) separated by a distance $2c$ is a given positive constant $2a$.

$$E = \{ P : |d(P, F_1) + d(P, F_2)| = 2a \}$$

With $F_1$ at $(-c, 0)$ and $F_2$ at $(c, 0)$ and setting $b = \sqrt{a^2 - c^2}$,

$$E = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$$
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The ellipse can also be given by a simple parametric form analogous to that of a circle, but with the $x$ and $y$ coordinates having different scalings,

$$x = a \cos t, \quad y = b \sin t, \quad t \in (0, 2\pi).$$

Note that $\cos^2 t + \sin^2 t = 1$. 
A hyperbola is the set of points $P$ in a plane that the difference of whose distances from two fixed points (the foci $F_1$ and $F_2$) separated by a distance $2c$ is a given positive constant $2a$.

$$H = \{ P : |d(P, F_1) - d(P, F_2)| = 2a \}$$

With $F_1$ at $(-c, 0)$ and $F_2$ at $(c, 0)$ and setting $b = \sqrt{c^2 - a^2}$, we have

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Hyperbolas

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$$H = \left\{ (x, y) : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\}$$
The right branch of a hyperbola can be parametrized by
\[ x = a \cosh t, \quad y = b \sinh t, \quad t \in (-\infty, \infty). \]

The left branch can be parametrized by
\[ x = -a \cosh t, \quad y = b \sinh t, \quad t \in (-\infty, \infty). \]

Note that \( \cosh t = \frac{1}{2} (e^t + e^{-t}) \), \( \sinh t = \frac{1}{2} (e^t - e^{-t}) \) and \( \cosh^2 t - \sinh^2 t = 1 \).
Hyperbolas: Hyperbolic Cosine and Hyperbolic Sine

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Another parametric representation for the right branch of the hyperbola is

\[ x = a \sec t, \quad y = b \tan t, \quad t \in (-\pi/2, \pi/2). \]

Parametric equations for the left branch is

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\[ x = a \sec t, \quad y = b \tan t, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \]

Parametric equations for the left branch is

\[ x = -a \sec t, \quad y = b \tan t, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \]
A lemniscate is the set of points \( P \) in a plane that the product of whose distances from two fixed points (the foci \( F_1 \) and \( F_2 \)) a distance \( 2c \) away is the constant \( c^2 \).

\[
R = \{ P : d(P, F_1) \cdot d(P, F_2) = c^2 \}
\]

With \( F_1 \) at \((-c, 0)\) and \( F_2 \) at \((c, 0)\),

\[
(x^2 + y^2)^2 = 2c^2(x^2 - y^2)
\]

Switching to polar coordinates gives

\[
r^2 = 2c^2 \cos 2\theta, \quad \theta \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \cup \left( \frac{3\pi}{4}, \frac{5\pi}{4} \right)
\]

The parametric equations for the lemniscate with \( a^2 = 2c^2 \) is

\[
x = \frac{a \cos t}{1 + \sin^2 t}, \quad y = \frac{a \sin t \cos t}{1 + \sin^2 t}, \quad t \in (0, 2\pi).
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A **lemniscate** is the set of points $P$ in a plane that the product of whose distances from two fixed points (the foci $F_1$ and $F_2$) a distance $2c$ away is the constant $c^2$.

$$R = \{ P : d(P, F_1) \cdot d(P, F_2) = c^2 \}$$

With $F_1$ at $(-c, 0)$ and $F_2$ at $(c, 0)$,

$$(x^2 + y^2)^2 = 2c^2(x^2 - y^2)$$

Switching to polar coordinates gives

$$r^2 = 2c^2 \cos 2\theta, \quad \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$$

The parametric equations for the lemniscate with $a^2 = 2c^2$ is

$$x = \frac{a \cos t}{1 + \sin^2 t}, \quad y = \frac{a \sin t \cos t}{1 + \sin^2 t}, \quad t \in (0, 2\pi).$$
Lemniscates (Ribbons): \( r^2 = a^2 \cos 2\theta \)

A lemniscate is the set of points \( P \) in a plane that the product of whose distances from two fixed points (the foci \( F_1 \) and \( F_2 \)) a distance \( 2c \) away is the constant \( c^2 \).

\[
R = \{ P : d(P, F_1) \cdot d(P, F_2) = c^2 \}
\]

With \( F_1 \) at \((-c, 0)\) and \( F_2 \) at \((c, 0)\),

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Outline

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