Lecture 16
Section 9.8 Arc Length and Speed

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What is the length of this curve?
Arc Length Formulas

Let \( C = \{(x(t), y(t)) : t \in I\} \).

The length of \( C \) is

\[
L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt
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d(P_{i-1}, P_i) = \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}
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= \sqrt{\left[\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}\right]^2 + \left[\frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}\right]^2} (t_i - t_{i-1})
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\[
= \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2} \Delta t_i
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\[
L(\gamma) = d(P_0, P_1) + \cdots + d(P_{i-1}, P_i) + \cdots + d(P_{n-1}, P_n)
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\[
= \sum_{i=1}^n \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2} \Delta t_i \to L(C) \quad \text{as } \Delta t_i \to 0.
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Definition

- We define the element of length \( ds \)
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- The total arc length is
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Arc Length and Speed Along a Plane Curve

**Parametrization by the Motion**

- Imaging an object moving along the curve $C$.
- Let $r(t) = (x(t), y(t))$ the position of the object at time $t$.
- The velocity of the object at time $t$ is $v(t) = r'(t) = (x'(t), y'(t))$.

**Arc Length and Speed Along a Plane Curve**

- The speed of the object at time $t$ is $v(t) = \|v(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$.
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- We have $ds = v(t) \, dt$. 

What is the length of this curve?
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Length of the Arc on the Graph of $y = f(x)$

The length of the arc on the graph from $a$ to $x$ is

$$s(x) = \int_{a}^{x} \sqrt{1 + [f'(t)]^2} \, dt.$$ 

$$\Rightarrow \, ds = \sqrt{1 + [f'(x)]^2} \, dx.$$ 

**Proof.**

Set $x(t) = t$, $y(t) = f(t)$, $t \in [a, b]$.

Since $x'(t) = 1$, $y'(t) = f'(t)$, then

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Length of \( y = f(x), \ x \in [a, b] \)

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Example

The length of the parabolic arc: \( f(x) = x^2, \ x \in [0, 1] \), is given

\[
\int_0^1 \sqrt{1 + [f'(x)]^2} \, dx = \int_0^1 \sqrt{1 + 4x^2} \, dx
\]

\[
= \left[ x \sqrt{\frac{1}{4} + x^2} + \frac{1}{4} \ln(x + \sqrt{\frac{1}{4} + x^2}) \right]_0^1 = \frac{1}{2} \sqrt{5} + \frac{1}{4} \ln(2 + \sqrt{5}).
\]
Length of the Arc on the Graph of $y = f(x)$

**Length of $y = f(x)$, $x \in [a, b]$**

The length of the arc on the graph from $a$ to $x$ is

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} \, dt.$$ 

$$\Rightarrow \quad ds = \sqrt{1 + [f'(x)]^2} \, dx.$$ 

**Example**

The length of the parabolic arc: $f(x) = x^2$, $x \in [0, 1]$, is given

$$\int_0^1 \sqrt{1 + [f'(x)]^2} \, dx = \int_0^1 \sqrt{1 + 4x^2} \, dx$$ 

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Length of the Arc on the Graph of $r = \rho(\theta)$

The length of the arc on the graph from $\alpha$ to $\theta$ is

$$s(\theta) = \int_{\alpha}^{\theta} \sqrt{[\rho(t)]^2 + [\rho'(t)]^2} \, dt.$$  

$\Rightarrow \quad ds(\theta) = \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2} \, d\theta$

Proof.

Set $x(t) = \rho(t) \cos t$, $y(t) = \rho(t) \sin t$, $t \in [\alpha, \beta]$.

Since $[x'(t)]^2 + [y'(t)]^2 = [\rho(t)]^2 + [\rho'(t)]^2$, then

$$s(\theta) = \int_{\alpha}^{\theta} \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt = \int_{\alpha}^{\beta} \sqrt{[\rho(t)]^2 + [\rho'(t)]^2} \, dt.$$
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The length of the arc on the graph from \( \alpha \) to \( \theta \) is

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\]

\[
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**Proof.**

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Spiral of Archimedes: \( r = \theta, \theta \geq 0 \)

- The length of the arc: \( r = \theta, \theta \in [0, 2\pi] \), is given

\[
\int_{0}^{2\pi} \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2} \, d\theta = \int_{0}^{2\pi} \sqrt{1 + \theta^2} \, d\theta
\]

\[
= \left[ \frac{1}{2} \theta \sqrt{1 + \theta^2} + \frac{1}{2} \ln(\theta + \sqrt{1 + \theta^2}) \right]_{0}^{2\pi} = \ldots
\]
**Length of the Arc on the Graph of** \( r = \rho(\theta) \)

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Spiral of Archimedes: $r = \theta$, $\theta \geq 0$

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Example: Circle of Radius $a$: $L = 2\pi a$

Circle in Polar Coordinates

$r = a$, $0 \leq \theta \leq 2\pi$

$$L = \int_0^{2\pi} \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2} \, d\theta = \int_0^{2\pi} \sqrt{a^2 + 0} \, d\theta = 2\pi a$$
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Example: Circle of Radius $a$: $L = 2\pi a$

Circle in Polar Coordinates

$$r = 2a \sin \theta, \quad 0 \leq \theta \leq \pi$$

Arc Length Example

$$L = \int_0^\pi \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2} \, d\theta$$

$$= \int_0^\pi \sqrt{[2a \sin \theta]^2 + [-2a \cos \theta]^2} \, d\theta = 2a \int_0^\pi \, d\theta = 2\pi a$$
Example: Circle of Radius \( a \): \( L = 2\pi a \)

Circle in Polar Coordinates

\[ r = 2a \sin \theta, \quad 0 \leq \theta \leq \pi \]

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Example: Limaçon

Limaçon: \( r = 1 - \cos \theta \)

The length of the cardioid: \( r = 1 - \cos \theta, \theta \in [0, 2\pi] \), is given

\[
\int_{0}^{2\pi} \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2} \, d\theta = 2 \int_{0}^{\pi} \sqrt{[\sin \theta]^2 + [1 - \cos \theta]^2} \, d\theta
\]

\[
= 2 \int_{0}^{\pi} \sqrt{2(1 - \cos \theta)} \, d\theta
\]

\[
= 2 \int_{0}^{\pi} 2 \sin \frac{1}{2} \theta \, d\theta = 8 \left[ -\cos \frac{1}{2} \theta \right]_{0}^{\pi} = 8
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Example: Logarithmic spiral \( r = ae^{b\theta} \)

A logarithmic spiral, equiangular spiral or growth spiral is a special kind of spiral curve which often appears in nature.

The polar equation of the curve is \( r = ae^{b\theta} \) or \( \theta = b^{-1} \ln(r/a) \).

The spiral has the property that the angle \( \phi \) between the tangent and radial line at the point \((r, \theta)\) is constant and \( \phi = \arctan b^{-1} \).

http://scienceblogs.com
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Logarithmic Spiral in Motion $r = ae^{-b\theta}$, $\theta \geq 0$

Spiral Motions

- The approach of a hawk to its prey. Their sharpest view is at an angle to their direction of flight.
- The approach of an insect to a light source. They are used to having the light source at a constant angle to their flight path.
- Starting at a point $P$ and moving inward along the spiral with the angle $\phi$. 

Let $a$ be the straight-line distance from $P$ to the origin. The spiral motion is described by 

$$\frac{dr}{d\theta} = -br,$$

$r(0) = a$, with $b = \cot \phi$.
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The polar equation of the path is

$$r = ae^{-b\theta}, \quad \theta \geq 0$$
Spiral Motions

- The approach of a hawk to its prey. Their sharpest view is at an angle to their direction of flight.

- The approach of an insect to a light source. They are used to having the light source at a constant angle to their flight path.

- Starting at a point $P$ and moving inward along the spiral with the angle $\phi$. Let $a$ be the straight-line distance from $P$ to the origin. The spiral motion is described by

\[
\frac{dr}{d\theta} = -b r, \quad r(0) = a, \quad \text{with } b = \cot \phi.
\]

The polar equation of the path is

\[
r = ae^{-b\theta}, \quad \theta \geq 0
\]
Logarithmic Spiral in Motion \( r = ae^{-b\theta}, \theta \geq 0 \)

Length of the Logarithmic Spiral: \( r = ae^{-b\theta}, \theta \geq 0 \)

- The length of the logarithmic spiral: \( r = e^{-\theta}, \theta \geq 0 \), is given by
  \[
  L(C) = \int_{0}^{\infty} \sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2} \, d\theta = \int_{0}^{\infty} \sqrt{[e^{-\theta}]^2 + [e^{-\theta}]^2} \, d\theta = \int_{0}^{\infty} \sqrt{2} \, e^{-\theta} \, d\theta = \sqrt{2} \left[ -e^{-\theta} \right]_{0}^{\infty} = \sqrt{2} \cdot \pi.
  \]

- The spiral motion \( r = ae^{-b\theta}, \theta \geq 0 \) circles the origin an unbounded number of times without reaching it; yet, the total distance covered on this path is finite:
  \[
  L(C) = \int_{0}^{\infty} ds = a / \cos(\phi), \quad \text{with } \phi = \cot^{-1} b.
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Four Bugs Chasing One Another

- Four bugs are at the corners of a square.
- They start to crawl clockwise at a constant rate, each moving toward its neighbor.
- At any instant, they mark the corners of a square. As the bugs get closer to the original square’s center, the new square they define rotates and diminishes in size.

\[ \frac{d}{d\theta} \theta = -r, \quad r(0) = \frac{1}{\sqrt{2}}. \]

The polar equation of the path is:

\[ r = \frac{1}{\sqrt{2}} e^{-\theta}, \quad \theta \geq 0. \]

The total distance covered on its path is:

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Each bug starts at a corner of the original (unit) square that is $\frac{1}{\sqrt{2}}$ away from the origin (i.e., center) and moves inward along the spiral with the angle $\phi = \frac{\pi}{4}$.

The spiral motion is described by $dr/d\theta = -r$, $r(0) = 1/\sqrt{2}$.

The polar equation of the path is $r = \frac{1}{\sqrt{2}} e^{-\theta}$, $\theta \geq 0$.

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Outline

- Arc Length
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  - Examples