Lecture 24 Section 11.4 Absolute and Conditional Convergence; Alternating Series
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1 Convergence Tests

Basic Series that Converge or Diverge

Basic Series that Converge

Geometric series: \( \sum x^k \), if \( |x| < 1 \)

\( p \)-series: \( \sum \frac{1}{k^p} \), if \( p > 1 \)

Basic Series that Diverge

Any series \( \sum a_k \) for which \( \lim_{k \to \infty} a_k \neq 0 \)

\( p \)-series: \( \sum \frac{1}{k^p} \), if \( p \leq 1 \)

Convergence Tests (1)

Basic Test for Convergence

Keep in Mind that, if \( a_k \to 0 \), then the series \( \sum a_k \) diverges; therefore there is no reason to apply any special convergence test.

Examples 1. \( \sum x^k \) with \( |x| \geq 1 \) (e.g., \( \sum (-1)^k \)) diverge since \( x^k \to 0 \). [1ex]

\( \sum \frac{k}{k+1} \) diverges since \( \frac{k}{k+1} \to 1 \neq 0 \). [1ex]

\( \sum \left( 1 - \frac{1}{k} \right)^k \) diverges since \( a_k = \left( 1 - \frac{1}{k} \right)^k \to e^{-1} \neq 0 \).

Convergence Tests (2)

Comparison Tests

Rational terms are most easily handled by basic comparison or limit comparison with \( p \)-series \( \sum 1/k^p \)

Basic Comparison Test
\[
\sum \frac{1}{2k^3 + 1} \text{ converges by comparison with } \sum \frac{1}{k^3} \sum \frac{k^3}{k^3 + 4k^4 + 7} \text{ converges by comparison with } \sum \frac{2}{k^3} \sum \frac{1}{k^3 - k^2} \text{ converges by comparison with } \sum \frac{3k + 1}{3k^3} \text{ diverges by comparison with } \sum \frac{1}{3(k + 1)} \sum \frac{1}{\ln(k + 6)} \text{ diverges by comparison with } \sum \frac{1}{k + 6} \]

**Limit Comparison Test**
\[
\sum \frac{1}{k^3 - 1} \text{ converges by comparison with } \sum \frac{1}{k^3} \sum \frac{3k^2 + 2k + 1}{k^3 + 1} \text{ diverges by comparison with } \sum \frac{3}{k} \sum \frac{5\sqrt{k} + 100}{2k^2 \sqrt{k} - 9 \sqrt{k}} \text{ converges by comparison with } \sum \frac{5}{2k^2} \]

**Convergence Tests (3)**

**Root Test and Ratio Test**
The root test is used only if powers are involved.

**Root Test**
\[
\sum \frac{k^2}{2k} \text{ converges: } (a_k)^{1/k} = \left(\frac{1}{2}\right)^{1/k} \to \frac{1}{2} \cdot 1 \sum \frac{1}{(\ln k)^k} \text{ converges: } (a_k)^{1/k} = \frac{1}{\ln k} \to 0 \sum \left(1 - \frac{1}{k}\right)^{k^2} \text{ converges: } (a_k)^{1/k} = \left(1 + \frac{(-1)^k}{k}\right)^k \to e^{-1} \]

**Convergence Tests (4)**

**Root Test and Ratio Test**
The ratio test is effective with factorials and with combinations of powers and factorials.

**Ratio Comparison Test**
\[
\sum \frac{k^2}{2k} \text{ converges: } \frac{a_{k+1}}{a_k} = \frac{1}{2} \cdot \frac{(k+1)^2}{k^2} \to \frac{1}{2} \sum \frac{1}{k^2} \text{ converges: } \frac{a_{k+1}}{a_k} = \frac{1}{k+1} \to 0
\]
\[
\sum \frac{k}{10} \text{ converges: } \frac{a_{k+1}}{a_k} = \frac{1}{10} \cdot \frac{k+1}{k} \to \frac{1}{10} \sum \frac{k}{k!} \text{ diverges: } \frac{a_{k+1}}{a_k} = (1 + \frac{1}{k})^k \to e
\]
\[
\sum \frac{2^k}{3^k - 2^k} \text{ converges: } \frac{a_{k+1}}{a_k} = 2 \cdot \frac{1 - (2/3)^k}{3 - 2(2/3)^k} \to 2 \cdot \frac{1}{3} \sum \frac{1}{\sqrt{k!}} \text{ converges: } \frac{a_{k+1}}{a_k} = \frac{1}{\sqrt{k+1}} \to 0
\]

2 Absolute Convergence

2.1 Absolute Convergence

Absolute Convergence
Absolute Convergence
A series \( \sum a_k \) is said to converge absolutely if \( \sum |a_k| \) converges.

\[
\text{if } \sum |a_k| \text{ converges, then } \sum a_k \text{ converges.}
\]
i.e., absolutely convergent series are convergent.

**Alternating \( p \)-Series with \( p > 1 \)**
\[
\sum \frac{(-1)^k}{k^p}, \ p > 1, \text{ converge absolutely because } \sum \frac{1}{k^p} \text{ converges.} \Rightarrow \\
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots \text{ converge absolutely.}
\]

Geometric Series with \( -1 < x < 1 \)
\[
\sum (-1)^k x^k, \ -1 < x < 1, \text{ converge absolutely because } \sum |x|^k \text{ converges.} \\
\Rightarrow 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} - \cdots \text{ converge absolutely.}
\]

Conditional Convergence
A series \( \sum a_k \) is said to converge conditionally if \( \sum a_k \) converges while \( \sum |a_k| \) diverges.

**Alternating \( p \)-Series with \( 0 < p \leq 1 \)**
\[
\sum \frac{(-1)^k}{k^p}, \ 0 < p \leq 1, \text{ converge conditionally because } \sum \frac{1}{k^p} \text{ diverges.} \Rightarrow \\
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots \text{ converge conditionally.}
\]

3 Alternating Series

**Alternating Series**
Let \( \{a_k\} \) be a sequence of positive numbers.
\[
\sum (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots
\]
is called an alternating series.

**Alternating Series Test**
Let \( \{a_k\} \) be a decreasing sequence of positive numbers.
\[
\text{If } a_k \to 0, \text{ then } \sum (-1)^k a_k \text{ converges.}
\]

**Alternating \( p \)-Series with \( p > 0 \)**
\[
\sum \frac{(-1)^k}{k^p}, \ p > 0, \text{ converge since } f(x) = \frac{1}{x^p} \text{ is decreasing, i.e., } f'(x) = -\frac{p}{x^{p+1}} > 0 \text{ for } \forall x > 0, \text{ and } \lim_{x \to \infty} f(x) = 0. \Rightarrow \\
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots \text{ converge conditionally.}
\]
Examples
\[ \sum \frac{(-1)^k}{2k+1} \text{ converge since } f(x) = \frac{1}{2x+1} \text{ is decreasing, i.e., } f'(x) = -\frac{2}{(2x+1)^2} > 0 \text{ for } \forall x > 0, \text{ and } \lim_{x \to \infty} f(x) = 0. \]

\[ \sum \frac{(-1)^k}{k^2 + 10} \text{ converge since } f(x) = \frac{x}{x^2 + 10} \text{ is decreasing, i.e., } f'(x) = -\frac{x^2 - 10}{(x^2 + 10)^2} > 0, \text{ for } \forall x > \sqrt{10}, \text{ and } \lim_{x \to \infty} f(x) = 0. \]

An Estimate for Alternating Series
Let \( \{a_k\} \) be a decreasing sequence of positive numbers that tends to 0 and let \( L = \sum_{k=0}^{\infty} (-1)^k a_k \). Then the sum \( L \) lies between consecutive partial sums \( s_n, s_{n+1} \),
\[ s_n < L < s_{n+1}, \text{ if } n \text{ is odd; } \quad s_{n+1} < L < s_n, \text{ if } n \text{ is even.} \]
and thus \( s_n \) approximates \( L \) to within \( a_{n+1} \)
\[ |L - s_n| < a_{n+1}. \]

Example
Find \( s_n \) to approximate \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots \) within \( 10^{-2} \).

Set \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \).
For \( |L - s_n| < 10^{-2} \), we want
\[ a_{n+1} = \frac{1}{(n+1)+1} < 10^{-2} \quad \Rightarrow \quad n + 2 > 10^2 \quad \Rightarrow \quad n > 98. \]
Then \( n = 99 \) and the 99th partial sum \( s_{100} \) is
\[ s_{99} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100} \approx 0.6882. \]
From the estimate
\[ |L - s_{99}| < a_{100} = \frac{1}{101} \approx 0.00991, \]
we conclude that
\[ s_{99} \approx 0.6882 < \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2 < 0.6981 \approx s_{100} \]

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Example
Find $s_n$ to approximate $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} \cdots$ within $10^{-2}$.

For $|L - s_n| < 10^{-2}$, we want
$$a_{n+1} = \frac{1}{(2(n+1)+1)!} < 10^{-2} \implies n \geq 1.$$ Then $n = 1$ and the 2nd partial sum $s_2$ is
$$s_1 = 1 - \frac{1}{3!} \approx 0.8333$$
From the estimate $|L - s_1| < a_2 = \frac{1}{5!} \approx 0.0083$.
we conclude that
$$s_1 \approx 0.8333 < \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} = \sin 1 < 0.8416 \approx s_2$$

4 Rearrangements

Why Absolute Convergence Matters: Rearrangements (1)

Rearrangement of Absolute Convergence Series
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \cdots = \frac{2}{3} \text{ absolutely}$$
$$\text{Rearrangement } 1 + \frac{1}{2^2} - \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^5} - \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} - \frac{1}{2^9} + \cdots = \frac{2}{3}$$

Theorem 2. All rearrangements of an absolutely convergent series converge absolutely to the same sum.

Why Absolute Convergence Matters: Rearrangements (2)

Rearrangement of Conditional Convergence Series
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2 \text{ conditionally}$$
$$\text{Rearrangement } 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \neq \ln 2$$

Multiply the original series by $\frac{1}{2}$
$$\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{10} + \cdots = \frac{1}{2} \ln 2$$
Adding the two series, we get the rearrangement
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2} \ln 2$$

Remark
• A series that is only conditionally convergent can be rearranged to converge to any number we please.

• It can also be arranged to diverge to $+\infty$ or $-\infty$, or even to oscillate between any two bounds we choose.

Outline

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