Lecture 25
Section 11.5 Taylor Polynomials in $x$; Taylor Series in $x$

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The $n$th Taylor polynomial at 0 for a function $f$ is

\[ P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \]

$P_n$ is the polynomial that has the same value as $f$ at 0 and the same first $n$ derivatives:

\[ P_n(0) = f(0), P'_n(0) = f'(0), P''_n(0) = f''(0), \cdots, P^{(n)}_n(0) = f^{(n)}(0). \]

Best Approximation

$P_n$ provides the best local approximation of $f(x)$ near 0 by a polynomial of degree $\leq n$.

\[ P_0(x) = f(0), \]
\[ P_1(x) = f(0) + f'(0)x, \]
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\(P_n\) provides the best local approximation of \(f(x)\) near 0 by a polynomial of degree \(\leq n\).

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Taylor Polynomials of the Exponential $f(x) = e^x$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad \cdots, \quad f^{(n)}(x) = e^x;$$

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \cdots, \quad f^{(n)}(0) = 1.$$
Taylor Polynomials of the Exponential $f(x) = e^x$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad \cdots, \quad f^{(n)}(x) = e^x; \quad f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \cdots, \quad f^{(n)}(0) = 1.$$
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\[ f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \cdots, \quad f^{(n)}(0) = 1. \]

Taylor Polynomials of $f(x) = e^x$

\[ P_0(x) = 1, \]
\[ P_1(x) = 1 + x, \]
\[ P_2(x) = 1 + x + \frac{x^2}{2!}, \]
\[ P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \]
\[ \vdots \]
\[ P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}. \]
Taylor Polynomials of the Exponential $f(x) = e^x$

$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$;

$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, $\cdots$, $f^{(n)}(x) = e^x$;
$f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, $\cdots$, $f^{(n)}(0) = 1$. 

Taylor Polynomials of $f(x) = e^x$

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$P_1(x) = 1 + x,$
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$\cdots$
$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$
Taylor Polynomials of the Exponential \( f(x) = e^x \)

\[
P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;
\]

\[
f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad \cdots, \quad f^{(n)}(x) = e^x; \\
f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \cdots, \quad f^{(n)}(0) = 1.
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Taylor Polynomials of the Exponential $f(x) = e^x$

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$$P_0(x) = 1,$$

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$$\vdots$$

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\[ P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \]

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\[ P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \]

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Taylor Polynomials of the Exponential $f(x) = e^x$

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P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;
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f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad \cdots, \quad f^{(n)}(x) = e^x;
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Taylor Polynomials of the Exponential $f(x) = e^x$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

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Taylor Polynomials of the Sine $f(x) = \sin x$

$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$;

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, \ldots

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{(4)}(0) = 0$, \ldots

Taylor Polynomials of $f(x) = \sin x$

$P_0(x) = 0$,

$P_1(x) = P_2(x) = x$,

$P_3(x) = P_4(x) = x - \frac{x^3}{3!}$,

$P_5(x) = P_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$,

$P_7(x) = P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$.
Taylor Polynomials of the Sine $f(x) = \sin x$

\[ P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \]

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, \ldots;

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Taylor Polynomials of $f(x) = \sin x$

\begin{align*}
P_0(x) &= 0, \\
P_1(x) &= P_2(x) = x, \\
P_3(x) &= P_4(x) = x - \frac{x^3}{3!}, \\
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P_7(x) &= P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.
\end{align*}
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P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;
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f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \cdots
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f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^{(4)}(0) = 0, \cdots
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Taylor Polynomials of the Sine $f(x) = \sin x$

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$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \ldots;$$

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Taylor Polynomials of $f(x) = \sin x$

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Jiwen He, University of Houston
Taylor Polynomials of the Sine $f(x) = \sin x$

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**Taylor’s Theorem**

If $f$ has $n + 1$ continuous derivatives on an open interval $I$ that contains 0, then for each $x \in I$,
\[ R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x - t)^n \, dt. \]

**Lagrange Formula for the Remainder**

For some number $c$ between 0 and $x$,
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Remainder Term

For each real $x$, $R_n(x) \to 0$ as $n \to \infty$.

Proof.

Let $J$ be the interval that joins 0 to $x$ and let $M = \max_{t \in J} e^t$.

Note that $f^{(n+1)}(t) = e^t$ for all $n$, then $\max_{t \in J} |f^{(n+1)}(t)| = M$.

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Note that $f^{(n+1)}(t) = e^t$ for all $n$, then $\max_{t \in J} |f^{(n+1)}(t)| = M$.

$|R_n(x)| \leq M \frac{|x|^{n+1}}{(n + 1)!} \to 0$ as $n \to \infty.$
Taylor Polynomials of the Exponential $f(x) = e^x$

$$P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}. $$

Taylor Polynomials of the Exponential $f(x) = e^x$

$$f(x) = e^x, \quad P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}. $$

Remainder Term

For each real $x$, $R_n(x) \to 0$ as $n \to \infty$.

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Let $J$ be the interval that joins 0 to $x$ and let $M = \max_{t \in J} e^t$. Note that $f^{(n+1)}(t) = e^t$ for all $n$, then $\max_{t \in J} |f^{(n+1)}(t)| = M$. Therefore,

$$|R_n(x)| \leq M \frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty.$$
Taylor Polynomials of the Sine $f(x) = \sin x$

\[ P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}. \]

\[ |R_n(x)| \leq \left( \max_{t \in J} |f^{(n+1)}(t)| \right) \frac{|x|^{n+1}}{(n+1)!}, \quad J = [0, x] \text{ or } [x, 0]. \]

Taylor Polynomials of the Sine $f(x) = \sin x$

\[ f(x) = \sin x, \quad P_7(x) = P_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \text{ and so on.} \]

Remainder Term

For each real $x$, $R_n(x) \to 0$ as $n \to \infty$.

\[ \forall k, \ f^{(k)}(t) = \pm \cos t \text{ or } \pm \sin t, \text{ then } \max_{t \in J} |f^{(n+1)}(t)| \leq 1. \]

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Remainder Term

For each real $x$, $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$.

$$\forall k, f^{(k)}(t) = \pm \cos t \text{ or } \pm \sin t, \text{ then } \max_{t \in J} |f^{(n+1)}(t)| \leq 1.$$  

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n + 1)!} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$
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$$|R_n(x)| \leq \left(\max_{t \in J} |f^{(n+1)}(t)|\right) \frac{|x|^{n+1}}{(n + 1)!}, \quad J = [0, x] \text{ or } [x, 0].$$

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For each real $x$, $R_n(x) \to 0$ as $n \to \infty$.

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$$P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.\quad \text{for some } c \in (0, x)$$

$$|R_n(x)| \leq \left( \max_{t \in J} |f^{(n+1)}(t)| \right) \frac{|x|^{n+1}}{(n+1)!}, \quad J = [0, x] \text{ or } [x, 0].$$

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Taylor Polynomials of the Sine $f(x) = \sin x$

Taylor Polynomials:

\[ P_n(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n; \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}. \]

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Taylor Series

Taylor Polynomial and the Remainder

If $f(x)$ is infinitely differentiable on interval $I$ containing 0, then

$$f(x) = P_n(x) + R_n(x), \quad \forall x \in I;$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!} x^n,$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad \text{or} \quad R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n \, dt.$$

Taylor Series

If $R_n(x) \to 0$ as $n \to \infty$, then $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \to f(x)$.

In this case, $f(x)$ can be expanded as a Taylor series in $x$ and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$
Taylor Series

Taylor Polynomial and the Remainder

If \( f(x) \) is infinitely differentiable on interval \( I \) containing 0, then
\[
f(x) = P_n(x) + R_n(x), \quad \forall x \in I;
\]
\[
P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!} x^n,
\]
\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad \text{or} \quad R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n \, dt.
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If \( R_n(x) \to 0 \) as \( n \to \infty \), then \( P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \to f(x) \).

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R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad \text{or} \quad R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x - t)^n \, dt.
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Taylor Series

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Taylor Polynomials

Taylor Series

Taylor Polynomial and the Remainder

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Taylor Series of the Exponential $f(x) = e^x$

$$f(x) = P_n(x) + R_n(x), \quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

If $\lim_{n \to \infty} R_n(x) \to 0$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n \to \infty} P_n(x).$$

Taylor Series of the Exponential $f(x) = e^x$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all real } x$$

Number $e$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$
Taylor Series of the Exponential $f(x) = e^x$

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Taylor Series of the Exponential $f(x) = e^x$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$ for all real $x$

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If \( \lim_{n \to \infty} R_n(x) \to 0 \), then \( f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n \to \infty} P_n(x) \).

**Taylor Series of the Exponential** $f(x) = e^x$

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Taylor Series of the Sine $f(x) = \sin x$

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Outline

- Taylor Polynomials
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  - Remainder Term

- Taylor Series
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  - Numerical Calculations