Lecture 2 Section 7.2 The Logarithm Function, Part I

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1 Definition and Properties of the Natural Log Function

1.1 Definition of the Natural Log Function

What We Do/Don't Know About $f(x) = x^r$?

We know that:

- For $r = n$ positive integer, $f(x) = x^n = \overbrace{x \cdot x \cdots x}^{n \text{ times}}$. To calculate 2^6, we do in our head or on a paper

  $$2 \times 2 \times 2 \times 2 \times 2 \times 2,$$

  but what does the computer actually do when we type

  ```
  2^6
  ```

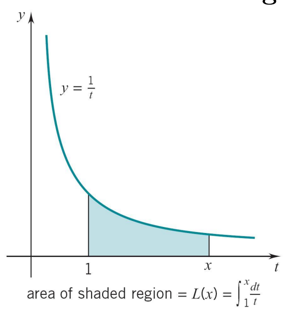
- For $r = 0$, $f(x) = x^0 = 1$.
- For $r = -n$, $f(x) = \left(\frac{1}{x}\right)^n$, $x \neq 0$. $\Rightarrow$ $x^{-1} = \frac{1}{x}$.
- For $r = \frac{p}{q}$ rational, $f(x) = y$, $x > 0$, where $y^q = x^p$. $f(x) = x^{\frac{1}{n}}$ is the inverse function of $g(x) = x^n$ for $x > 0$. $\Rightarrow$ $g \circ f(x) = \left(x^{\frac{1}{n}}\right)^n = x$.
- Properties (r and s rational)

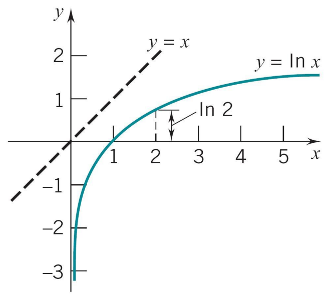
  $$x^{r+s} = x^r \cdot x^s, \quad x^{r \cdot s} = \left(x^r\right)^s,$$
  $$\frac{d}{dx} x^r = r x^{r-1}, \quad \int x^r \, dx = \frac{1}{r+1} x^{r+1} + C, \quad r \neq -1.$$

We DO NOT know *yet* that:

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = ? \qquad \text{and} \qquad x^r = ? \text{ for } r \text{ real}.$$

What is the Natural Log Function?





Definition 1. The function

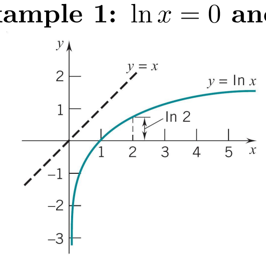
$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0,$$

is called the *natural logarithm function.*

- $\ln 1 = 0$.
- $\ln x < 0$ for $0 < x < 1$, $\ln x > 0$ for $x > 1$.
- $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0 \quad \Rightarrow \quad \ln x$ is increasing.
- $\frac{d^2}{dx^2}(\ln x) = -\frac{1}{x^2} < 0 \quad \Rightarrow \quad \ln x$ is concave down.

1.2 Examples

Example 1: $\ln x = 0$ **and** $(\ln x)' = 1$ **at** $x = 1$



Exercise 7.2.23
Show that

$$\lim_{x\to 1} \frac{\ln x}{x-1} = 1.$$

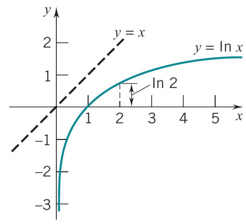
Proof.

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\ln x - \ln 1}{x-1} = \frac{d}{dx}(\ln x)\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1.$$

$\square$

The limit has the *indeterminate form* $\left(\frac{0}{0}\right)$ and is interpreted here in terms of the *derivative* of $\ln x$.

Example 2: $\ln x$ **and** $x - 1$



Exercise 7.2.24(a)
Show that

$$\frac{x-1}{x} \leq \ln x \leq x - 1, \quad \forall x > 0. \tag{1}$$

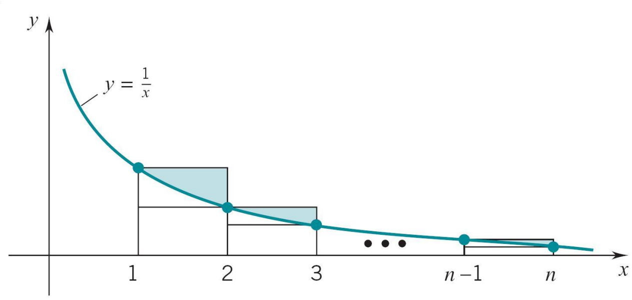
Proof.

- By the mean-value theorem, $\exists c$ between 1 and x s.t.

$$\ln x = \int_1^x \frac{1}{t}\, dt = \frac{1}{c}(x-1).$$

- If $x > 1$, then $\frac{1}{x} < \frac{1}{c} < 1$ and $x - 1 > 0$ so (1) holds.
- If $0 < x < 1$, then $1 < \frac{1}{c} < \frac{1}{x}$ and $x - 1 < 0$ so (1) holds. $\square$

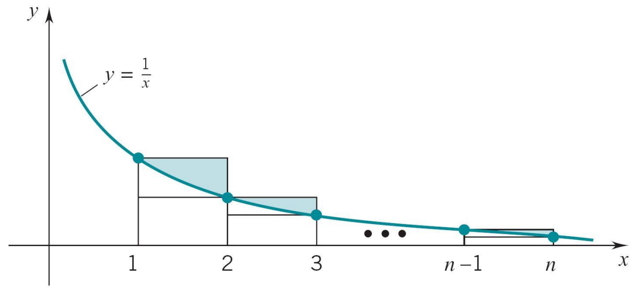
Example 3: $\ln n$ **and Harmonic Number**



Exercise 7.2.25(a)
Show that for $n \geq 2$

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

Example 3: $\ln n$ **and Harmonic Number**

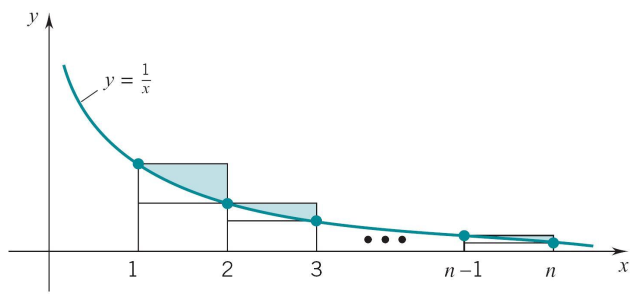


Proof. Let $P = \{1, 2, \cdots, n\}$ be a partition of $[1, n]$. Then

$$L_f(P) = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{1}{t}\,dt < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} = U_f(P).$$

□

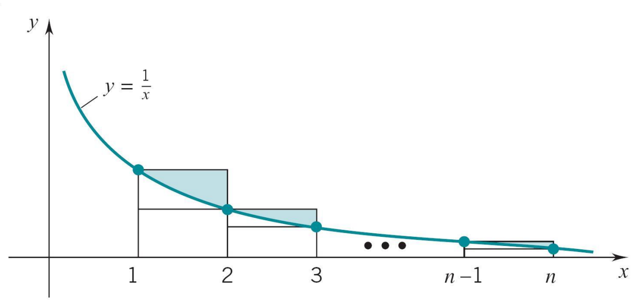
Example 4: Euler's Constant γ



Exercise 7.2.25(c)
Show that

$$\frac{1}{2} < \gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln n\right) < 1.$$

Example 4: Euler's Constant γ

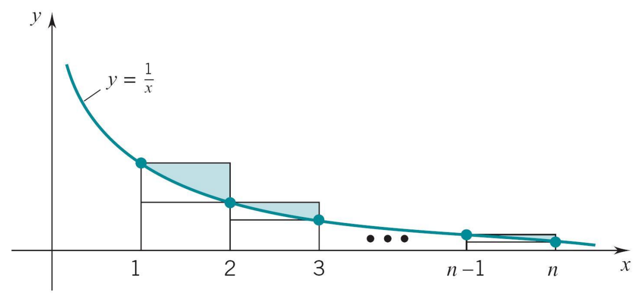


Proof.

- The sum of the shaded areas is given by

$$S_n = U_f(P) - \int_1^n \frac{1}{t}\,dt = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln n.$$

Example 4: Euler's Constant γ

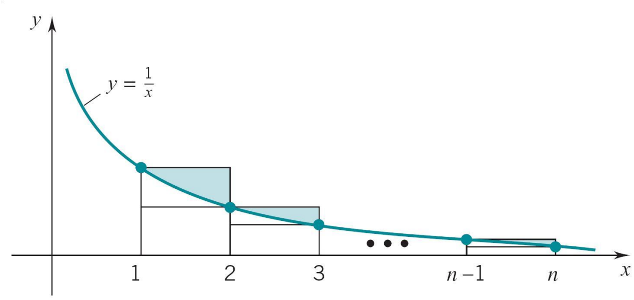


Proof. (cont.)

- The sum of the areas of the triangles formed by connecting the points $(1,1)$, $\cdots$, $(n, \frac{1}{n})$ is

$$T_n = \frac{1}{2} \cdot 1 \left[\left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \right] = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

Example 4: Euler's Constant γ

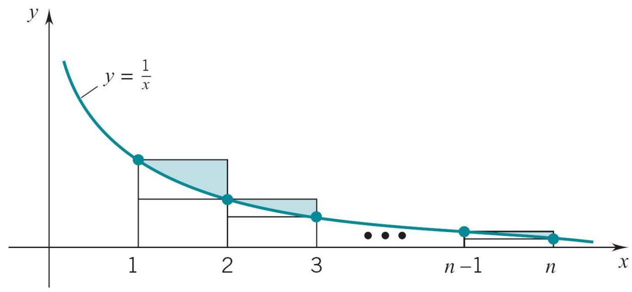


Proof. (cont.)

- The sum of the areas of the indicated rectangles is

$$R_n = 1 \left[\left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \right] = 1 - \frac{1}{n}.$$

Example 4: Euler's Constant γ



Proof. (cont.)

- Since $T_n < S_n < R_n$,

$$\frac{1}{2}\left(1-\frac{1}{n}\right) < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}-\ln n < 1-\frac{1}{n}.$$

Letting $n \to \infty$ we have $\frac{1}{2} < \gamma < 1$.

1.3 Algebraic Properties of the Natural Log Function

Basic Property: $\ln(xy) = \ln x + \ln y$

Lemma 2.

$$\ln(xy) = \ln x + \ln y, \quad x>0, y>0.$$

Proof.

- Left side:

$$\frac{d}{dx}\ln(xy) = \frac{1}{xy}\,y = \frac{1}{x}.$$

- Right side:

$$\frac{d}{dx}\big(\ln x + \ln y\big) = \frac{1}{x}.$$

- Then

$$\ln(xy) = \ln x + \ln y + C$$

for some constant C. At $x = 1$, both sides take the same value of $\ln y$, thus $C = 0$. □

Basic Property: $\ln x^r = r \ln x$

Lemma 3.

$$\ln x^r = r\ln x \quad (r \text{ rational}).$$

Proof.

- Left side:

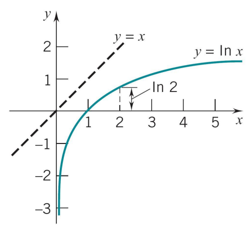
$$\frac{d}{dx}\ln x^r = \frac{1}{x^r}\frac{d}{dx}x^r = \frac{1}{x^r}r\,x^{r-1} = r\frac{1}{x}.$$

- Right side:
$$\frac{d}{dx}\big(r\,\ln x\big) = r\frac{1}{x}.$$
- Then
$$\ln x^r = r\ln x + C$$
for some constant C. At $x = 1$, both sides are zero, thus $C = 0$. $\square$

2 Range and Limits of the Natural Log Function

2.1 Range of the Natural Log Function

Range $= (-\infty, \infty)$

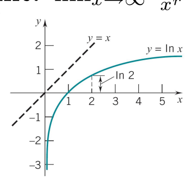


Theorem 4. *The log function* $\ln x$ *has range* $(-\infty, \infty)$ *and*
$$\lim_{x\to 0^+} \ln x = -\infty, \quad \lim_{x\to\infty} \ln x = \infty.$$

Proof.
- Let $M > 0$ arbitrary in $\mathbb{R}$. Since $\ln 2 > 0$, $\exists n \in \mathbb{N}$ s.t.
$$n\ln 2 > M, \quad -n\ln 2 < -M.$$
- Since $n\ln 2 = \ln(2^n)$ and $-n\ln 2 = \ln(2^{-n})$,
$$\ln(2^n) > M, \quad \ln(2^{-n}) < -M.$$
$\square$

2.2 Limits of the Natural Log Function

Limit: $\lim_{x\to\infty} \frac{\ln x}{x^r}$



Theorem 5.
$$\lim_{x\to\infty} \frac{\ln x}{x^r} = 0 \quad \textit{for any } r > 0.$$
$\ln x$ *grows slower than any positive power as* $x \to \infty$.

Proof. • Choose a rational number p s.t. $1-r<p<1$. For $x>1$,

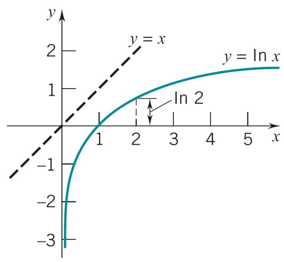
$$\ln x = \int_1^x \frac{1}{t}\,dt < \int_1^x \frac{1}{t^p}\,dt = \frac{1}{1-p}\,t^{1-p}\Big|_1^x = \frac{1}{1-p}\big(x^{1-p}-1\big).$$

• Then

$$0 < \frac{\ln x}{x^r} < \frac{1}{1-p}\,\frac{x^{1-p}-1}{x^r} = \frac{1}{1-p}\big(x^{1-p-r}-x^{-r}\big)$$

Use the pinching theorem to take the limit as $x\to\infty$. □

Limit: $\lim_{x\to 0^+} x^r \ln x$



Corollary 6.

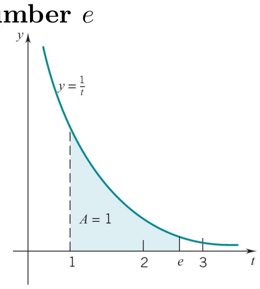
$$\lim_{x\to 0^+} x^r \ln x = 0 \quad \textit{for any } r>0.$$

Proof. Let $y=x^{-1}$. Then

$$\lim_{x\to 0^+} x^r \ln x = \lim_{y\to\infty} y^{-r} \ln y^{-1} = -\lim_{y\to\infty} \frac{\ln y}{y^r} = 0.$$ □

3 Number e

Number e



Definition 7. The number e is defined by

$$\ln e = 1$$

i.e., the unique number at which $\ln x = 1$.

Theorem 8.

$$\ln e^r = r \quad \textit{for any rational number } r.$$

Proof.

$$\ln e^r = r \ln e = r$$

□

Quiz

Quiz

1.	$\ln 1 = ?$:	(a) -1,	(b) 0,	(c) 1.
2.	$\ln e = ?$:	(a) 0,	(b) 1,	(c) e.
3.	$\lim_{x \to 0^+} \ln x = ?$:	(a) $-\infty$,	(b) 0,	(c) ∞.
4.	$\lim_{x \to \infty} \ln x = ?$:	(a) $-\infty$,	(b) 0,	(c) ∞.

Outline

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