# Math 2331 – Linear Algebra

## 1.2 Row Reduction and Echelon Forms

**Jiwen He**

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math2331

# 1.2 Row Reduction and Echelon Forms

- Echelon Form and Reduced Echelon Form
  - Uniqueness of the Reduced Echelon Form
  - Pivot and Pivot Column
  - Row Reduction Algorithm
    - Reduce to Echelon Form (Forward Phase)
    - then to REF (Backward Phase)
- Solutions of Linear Systems
  - Basic Variables and Free Variable
  - Parametric Descriptions of Solution Sets
  - Final Steps in Solving a Consistent Linear System
    - Back-Substitution
  - General Solutions
- Existence and Uniqueness Theorem
  - Using Row Reduction to Solve Linear Systems
  - Consistency Questions

# Echelon Forms

## Echelon Form (or **Row Echelon Form**)

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

## Examples (Echelon forms)

$$(a)\ \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (b)\ \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

$$(c)\ \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

# Reduced Echelon Form

## Reduced Echelon Form

Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

## Example (Reduced Echelon Form)

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

## Theorem (Uniqueness of the Reduced Echelon Form)

*Each matrix is row-equivalent to one and only one reduced echelon matrix.*

# Pivots

## Important Terms

- **pivot position:** a position of a leading entry in an echelon form of the matrix.
- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.
- **pivot column:** a column that contains a pivot position.

(See the Glossary at the back of the textbook.)

# Reduced Echelon Form: Examples

## Example (Row reduce to echelon form and locate the pivots)

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

**Solution**

pivot

$\swarrow$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\uparrow$

pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Possible Pivots:

# Reduced Echelon Form: Examples (cont.)

## Example (Row reduce to echelon form (cont.))

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Original Matrix: } \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\begin{matrix} & \uparrow & \uparrow & & \uparrow \\ \text{pivot columns:} & 1 & 2 & & 4 \end{matrix}$$

## Note

There is no more than one pivot in any row. There is no more than one pivot in any column.

# Reduced Echelon Form: Examples (cont.)

## Example (Row reduce to echelon form and then to REF)

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

# Reduced Echelon Form: Examples (cont.)

**Example (Row reduce to echelon form and then to REF (cont.))**

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{(echelon form)}$$

# Reduced Echelon Form: Examples (cont.)

## Example (Row reduce to echelon form and then to REF (cont.))

**Final step to create the reduced echelon form:**

Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Solutions of Linear Systems

## Important Terms

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **free variable:** all nonbasic variables.

## Example (Solutions of Linear Systems)

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad +6x_2 \quad \quad +3x_4 \quad &= 0 \\ x_3 \quad -8x_4 \quad &= 5 \\ x_5 &= 7 \end{aligned}$$

pivot columns:

basic variables:

free variables:

# Solutions of Linear Systems (cont.)

## Final Step in Solving a Consistent Linear System

After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations, *Solve each equation for the basic variable in terms of the free variables (if any) in the equation.*

## Example (General Solutions of Linear Systems)

$$\begin{array}{lllll} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

(general solution)

## Warning

Use only the reduced echelon form to solve a system.

# General Solutions of Linear Systems

## General Solution

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.)

## Example (General Solutions of Linear Systems (cont.))

$$x_1 = -6x_2 - 3x_4$$
$$x_2 \text{ is free}$$
$$x_3 = 5 + 8x_4$$
$$x_4 \text{ is free}$$
$$x_5 = 7$$

The above system has **infinitely many solutions.** Why?

# Existence and Uniqueness Questions

## Example (Existence and Uniqueness Questions)

$$\begin{bmatrix} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{bmatrix}$$

In an earlier example, we obtained the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is **consistent**. **Free variables:** $x_3$ and $x_4$.

**Consistent system with free variables** $\Longrightarrow$ **infinitely many solutions.**

# Existence and Uniqueness Questions

## Example (Existence and Uniqueness Questions)

$$\begin{array}{rrcr} 3x_1 & +4x_2 & = & -3 \\ 2x_1 & +5x_2 & = & 5 \\ -2x_1 & -3x_2 & = & 1 \end{array} \rightarrow \begin{bmatrix} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

**Consistent system, no free variables** $\Longrightarrow$ **unique solution.**

# Existence and Uniqueness Theorem

## Theorem (Existence and Uniqueness)

1. *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form*

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{ (where } b \text{ is nonzero).}$$

2. *If a linear system is consistent, then the solution contains either*
   - *a unique solution (when there are no free variables) or*
   - *infinitely many solutions (when there is at least one free variable).*

# Using Row Reduction to Solve Linear Systems

## Using Row Reduction to Solve Linear Systems

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.

# Consistency Questions

## Example (a)

What is the largest possible number of pivots a $4 \times 6$ matrix can have? Why?

## Example (b)

What is the largest possible number of pivots a $6 \times 4$ matrix can have? Why?

# Consistency Questions (cont.)

## Example (c)

How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

## Example (d)

Suppose the coefficient matrix corresponding to a linear system is $4 \times 6$ and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?