Math 2331 – Linear Algebra

1.4 The Matrix Equation $\mathbf{Ax} = \mathbf{b}$

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1.4 The Matrix Equation $\mathbf{Ax} = \mathbf{b}$

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- Matrix Equation
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Matrix-Vector Multiplication

Key Concepts to Master

Linear combinations can be viewed as
a matrix-vector multiplication.

Matrix-Vector Multiplication

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**. i.e.,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Matrix-Vector Multiplication: Examples

Example

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6 \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

Matrix-Vector Multiplication: Examples

Example

Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is a 3×1 vector.

$$\begin{bmatrix} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{bmatrix}$$

Solution: Corresponding system of equations (fill-in)

Vector Equation:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \qquad \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \qquad \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Matrix equation (fill-in):

Matrix Equation

Three Equivalent Ways of Viewing a Linear System

1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$; **or**
3. as a matrix equation $A\mathbf{x} = \mathbf{b}$.

Useful Fact

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a

_________ _______________ of the columns of A.

Matrix Equation: Theorem

Theorem

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbf{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Matrix Equation: Example

Example

Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is _______ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

Matrix Equation: Example (cont.)

$$A = \begin{array}{c} \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array} \end{array}$$

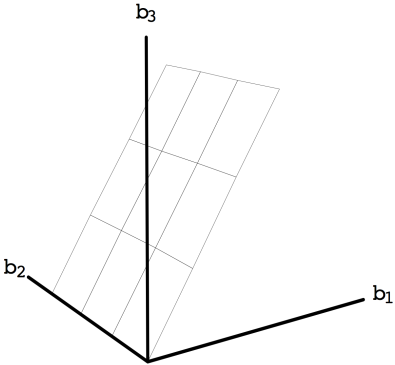
The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_3 + x_3\mathbf{a}_3 = \mathbf{b}$$

if and only if $b_3 - 2b_1 = 0$.



Columns of A span a plane in $\mathbf{R}^3$ through $\mathbf{0}$

Instead, if *any* $\mathbf{b}$ in $\mathbf{R}^3$ (not just those lying on a particular line or in a plane) can be expressed as a linear combination of the columns of A, then we say that the columns of A span $\mathbf{R}^3$.

Matrix Equation: Span $\mathbf{R}^m$

Definition

We say that **the columns of** $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$ **span** $\mathbf{R}^m$ if *every* vector $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$ (i.e. $\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} = \mathbf{R}^m$).

Theorem (4)

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. *For each $\mathbf{b}$ in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*
2. *Each $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of the columns of A.*
3. *The columns of A span $\mathbf{R}^m$.*
4. *A has a pivot position in every row.*

Matrix Equation: Proof of Theorem

Proof (outline): Statements (a), (b) and (c) are logically equivalent.

To complete the proof, we need to show that (a) is true when (d) is true and (a) is false when (d) is false.

Suppose (d) is _____________. Then row-reduce the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$:

$$[A \quad \mathbf{b}] \sim \cdots \sim [U \quad \mathbf{d}]$$

and each row of U has a pivot position and so there is no pivot in the last column of $[U \quad \mathbf{d}]$.

So (a) is _____________.

Matrix Equation: Proof of Theorem (cont.)

Now suppose (d) is ____________. Then the last row of $[U \quad \mathbf{d}]$ contains all zeros.

Suppose $\mathbf{d}$ is a vector with a 1 as the last entry. Then $[U \quad \mathbf{d}]$ represents an inconsistent system.

Row operations are reversible: $[U \quad \mathbf{d}] \sim \cdots \sim [A \quad \mathbf{b}]$

$\Longrightarrow [A \quad \mathbf{b}]$ is inconsistent also. So (a) is ___________. ■

Matrix Equation: Example

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $\mathbf{b}$?

Solution: A has only _____ columns and therefore has at most _____ pivots.

Since A does not have a pivot in every _______, $A\mathbf{x} = \mathbf{b}$

is ________ ____________________ for all possible $\mathbf{b}$, according to Theorem 4.

Matrix Equation: Example

Example

Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbf{R}^3$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix} \sim$$

(no pivot in row 2)

By Theorem 4, the columns of A

____.

Another method for computing $A\mathbf{x}$: Row-Vector Rule

Another method for computing $A\mathbf{x}$: Row-Vector Rule

Read Example 4 on page 37 through Example 5 on page 38 to learn this rule for computing the product $A\mathbf{x}$.

Theorem

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^n$, and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
2. $A(c\mathbf{u}) = cA\mathbf{u}$.