Math 2331 – Linear Algebra

3.1 Introduction to Determinants

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3.1 Introduction to Determinants

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Introduction to Determinants

Notation: A_{ij} is the matrix obtained from matrix A by deleting the ith row and jth column of A.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Recall that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det [a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Determinants: Example

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution

$$\det A = 1 \det \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \underline{\hspace{8cm}} = \underline{\hspace{1.5cm}}$$

Common notation: $\det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

Determinants: Cofactor Expansion

Cofactor

The **(i, j)-cofactor** of A is the number C_{ij} where

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Example (Cofactor Expansion)

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

Cofactor Expansion: Theorem

Theorem (Cofactor Expansion)

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(expansion across row i)

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(expansion down column j)

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Cofactor Expansion: Example

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

Cofactor Expansion: Example

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

Solution:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0 \begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 2 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

Triangular Matrices

Method of cofactor expansion is not practical for large matrices - see Numerical Note on page 167.

Triangular Matrices

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

Theorem

If A is a triangular matrix, then $\det A$ is the product of the main diagonal entries of A.

Triangular Matrices: Example

Example

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \rule{4cm}{0.4pt} = -24$$