Math 2331 – Linear Algebra

4.2 Null Spaces, Column Spaces, & Linear Transformations

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The null space of an \( m \times n \) matrix \( A \), written as Nul \( A \), is the set of all solutions to the homogeneous equation \( Ax = 0 \).

\[
\text{Nul} \ A = \{ \mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = 0 \} \quad \text{(set notation)}
\]

**Theorem (2)**

The null space of an \( m \times n \) matrix \( A \) is a subspace of \( \mathbb{R}^n \). Equivalently, the set of all solutions to a system \( Ax = 0 \) of \( m \) homogeneous linear equations in \( n \) unknowns is a subspace of \( \mathbb{R}^n \).

**Proof:** Nul \( A \) is a subset of \( \mathbb{R}^n \) since \( A \) has \( n \) columns. Must verify properties a, b and c of the definition of a subspace.

**Property (a)** Show that \( \mathbf{0} \) is in Nul \( A \). Since ________, \( \mathbf{0} \) is in ________.
Null Space (cont.)

**Property (b)** If \( u \) and \( v \) are in Nul \( A \), show that \( u + v \) is in Nul \( A \).
Since \( u \) and \( v \) are in Nul \( A \),

\[
\text{________ and ________}. 
\]

Therefore

\[
A (u + v) = \text{________ } + \text{________ } = \text{________ } + \text{________ } = \text{______}_. 
\]

**Property (c)** If \( u \) is in Nul \( A \) and \( c \) is a scalar, show that \( cu \) in Nul \( A \):

\[
A (cu) = \text{___}A (u) = c0 = 0. 
\]

Since properties a, b and c hold, \( A \) is a subspace of \( \mathbb{R}^n \).
Solving \( Ax = 0 \) yields an *explicit description* of Nul \( A \).
Example

Find an explicit description of Nul $A$ where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$$

Solution: Row reduce augmented matrix corresponding to $Ax = 0$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$
Null Space: Example (cont.)

\[ A = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 1 \end{bmatrix} \]

Then

\[ \text{Nul } A = \text{span}\{u, v, w\} \]
Observations:
1. Spanning set of Nul $A$, found using the method in the last example, is automatically linearly independent:

$$
\begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0
\end{bmatrix} c_1 +
\begin{bmatrix}
  -13 \\
  0 \\
  6 \\
  1
\end{bmatrix} c_2 +
\begin{bmatrix}
  -33 \\
  0 \\
  15 \\
  1
\end{bmatrix} c_3 =
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\Rightarrow
\begin{align*}
  c_1 &= \text{_____} \\
  c_2 &= \text{_____} \\
  c_3 &= \text{_____}
\end{align*}
$$

2. If Nul $A \neq \{0\}$, the number of vectors in the spanning set for Nul $A$ equals the number of free variables in $Ax = 0$. 
Column Space

The **column space** of an $m \times n$ matrix $A$ (Col $A$) is the set of all linear combinations of the columns of $A$.

If $A = [a_1 \ldots a_n]$, then

$$\text{Col } A = \text{Span}\{a_1, \ldots, a_n\}$$

**Theorem (3)**

*The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.***

Why? (Theorem 1, page 194)

Recall that if $Ax = b$, then $b$ is a linear combination of the columns of $A$. Therefore

$$\text{Col } A = \{b : b = Ax \text{ for some } x \text{ in } \mathbb{R}^n\}$$
Column Space: Example

Example

Find a matrix $A$ such that $W = \text{Col } A$ where

$$W = \left\{ \begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}.$$  

Solution:

$$\begin{bmatrix} x - 2y \\ 3y \\ x + y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$
Therefore

\[ A = \begin{bmatrix} \end{bmatrix}. \]

By Theorem 4 (Chapter 1),

The column space of an \( m \times n \) matrix \( A \) is all of \( \mathbb{R}^m \) if and only if the equation \( Ax = b \) has a solution for each \( b \) in \( \mathbb{R}^m \).
The Contrast Between Nul $A$ and Col $A$

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of $A$ is a subspace of $\mathbb{R}^k$ where $k =\ldots$.

(b) The null space of $A$ is a subspace of $\mathbb{R}^k$ where $k =\ldots$.

(c) Find a nonzero vector in Col $A$. (There are infinitely many possibilities.)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \ldots \end{bmatrix}$$
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The Contrast Between Nul $A$ and Col $A$ (cont.)

Example (cont.)

(d) Find a nonzero vector in Nul $A$. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
2 & 4 & 7 & 0 \\
3 & 6 & 10 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

row reduces to

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

$x_1 = -2x_2$

$x_2$ is free $\implies$ let $x_2 = ___ \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$

Contrast Between Nul $A$ and Col $A$ where $A$ is $m \times n$ (see page 204)
A \textbf{subspace} of a vector space $V$ is a subset $H$ of $V$ that has three properties:

a. The zero vector of $V$ is in $H$.

b. For each $u$ and $v$ in $H$, $u + v$ is in $H$. (In this case we say $H$ is closed under vector addition.)

c. For each $u$ in $H$ and each scalar $c$, $cu$ is in $H$. (In this case we say $H$ is closed under scalar multiplication.)

If the subset $H$ satisfies these three properties, then $H$ itself is a vector space.
Null Spaces & Column Spaces: Review (cont.)

Theorem (1, 2 and 3 in Sections 4.1 & 4.2)

- If $v_1, \ldots, v_p$ are in a vector space $V$, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of $V$.

- The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$.

- The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$. 
Null Spaces & Column Spaces: Examples

**Example**

Determine whether each of the following sets is a vector space or provide a counterexample.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$

**Solution:** Since

\[ \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \ldots \\ \ldots \end{bmatrix} \]

is not in $H$, $H$ is not a vector space.
Example

(b) \( V = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{align*} x - y &= 0 \\ y + z &= 0 \end{align*} \} \)

Solution: Rewrite

\[ x - y = 0 \]
\[ y + z = 0 \]

as

\[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

So \( V = \text{Nul } A \) where \( A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \). Since \( \text{Nul } A \) is a subspace of \( \mathbb{R}^2 \), \( V \) is a vector space.
Example

(c) \( S = \left\{ \begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\} \)

One Solution: Since

\[
\begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},
\]

\[
S = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\};
\]

therefore \( S \) is a vector space by Theorem 1.
Another Solution: Since

\[
\begin{bmatrix}
  x + y \\
  2x - 3y \\
  3y
\end{bmatrix}
= x \begin{bmatrix}
  1 \\
  2 \\
  0
\end{bmatrix}
+ y \begin{bmatrix}
  1 \\
  -3 \\
  3
\end{bmatrix},
\]

\[S = \text{Col } A \quad \text{where } A = \begin{bmatrix}
  1 & 1 \\
  2 & -3 \\
  0 & 3
\end{bmatrix};\]

therefore \(S\) is a vector space, since a column space is a vector space.
Linear Transformation

A **linear transformation** $T$ from a vector space $V$ into a vector space $W$ is a rule that assigns to each vector $\mathbf{x}$ in $V$ a unique vector $T(\mathbf{x})$ in $W$, such that

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in $V$;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in $V$ and all scalars $c$.

Kernal and Range

The **kernel** (or **null space**) of $T$ is the set of all vectors $\mathbf{u}$ in $V$ such that $T(\mathbf{u}) = \mathbf{0}$. The **range** of $T$ is the set of all vectors in $W$ of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in $V$.

So if $T(\mathbf{x}) = A\mathbf{x}$, $\text{col } A = \text{range of } T$. 