Math 2331 – Linear Algebra

6.1-6.3 Orthogonality
Key Exercises

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6.1 Inner Product, Length & Orthogonality
Key Exercises 19–20, 27–31

- The general material on orthogonal complements is essential for later work.
- Key Exercises: 19–20, 27–31. Exercies 27–31 concern facts that are needed later

All vectors are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

19. a. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

b. For any scalar c, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.

c. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

d. For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.

e. If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^\perp$.

All vectors are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

20. a. $\mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} = 0$.

b. For any scalar c, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.

c. If $\mathbf{x}$ is orthogonal to every vector in a subspace W, then $\mathbf{x}$ is in $W^{\perp}$.

d. If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

e. For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.

27. Suppose a vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to the vector $\mathbf{u} + \mathbf{v}$.

28. Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$. [*Hint:* An arbitrary $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to such a vector $\mathbf{w}$.]

29. Let $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, for $1 \leq j \leq p$, then $\mathbf{x}$ is orthogonal to every vector in W.

30. Let W be a subspace of $\mathbb{R}^n$, and let $W^\perp$ be the set of all vectors orthogonal to W. Show that $W^\perp$ is a subspace of $\mathbb{R}^n$ using the following steps.

a. Take $\mathbf{z}$ in $W^\perp$, and let $\mathbf{u}$ represent any element of W. Then $\mathbf{z} \cdot \mathbf{u} = 0$. Take any scalar c and show that $c\mathbf{z}$ is orthogonal to $\mathbf{u}$. (Since $\mathbf{u}$ was an arbitrary element of W, this will show that $c\mathbf{z}$ is in $W^\perp$.)

b. Take $\mathbf{z}_1$ and $\mathbf{z}_2$ in $W^\perp$, and let $\mathbf{u}$ be any element of W. Show that $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to $\mathbf{u}$. What can you conclude about $\mathbf{z}_1 + \mathbf{z}_2$? Why?

c. Finish the proof that $W^\perp$ is a subspace of $\mathbb{R}^n$.

31. Show that if $\mathbf{x}$ is in both W and $W^{\perp}$, then $\mathbf{x} = \mathbf{0}$.

6.2 Orthogonal Sets
Key Exercises 13–14, 23–24, 26–30

- The term orthogonal matrix applies only to certain square matrices.
- Key Exercises: 13–14, 23–24, 26–30. Exercies 13–14 prepare for Section 6.3.

13. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

14. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span $\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

All vectors are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

23. a. Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

b. If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c. If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d. A matrix with orthonormal columns is an orthogonal matrix.

e. If L is a line through $\mathbf{0}$ and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

All vectors are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

24. a. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.

c. If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.

d. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

e. An orthogonal matrix is invertible.

26. Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

27. Let U be a square matrix with orthonormal columns. Explain why U is invertible. (Mention the theorems you use.)

28. Let U be an $n \times n$ orthogonal matrix. Show that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

29. Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix. [That is, explain why UV is invertible and its inverse is $(UV)^T$.]

30. Let U be an orthogonal matrix, and construct V by interchanging some of the columns of U. Explain why V is an orthogonal matrix.

6.3 Orthogonal Projections
Key Exercises 19–24

- Theorem 9 is needed for the Gram-Schmidt process.
- Key Exercises: 19–24. Exercies 19–20 lead naturally into the Gram-Schmidt process.

19. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal but that $\mathbf{u}_3$ is not orthogonal to $\mathbf{u}_1$ or $\mathbf{u}_2$. It can be shown that $\mathbf{u}_3$ is not in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Use this fact to construct a nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

20. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be as in Exercise 19, and let $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{u}_4$ is not in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Use this fact to construct a nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

All vectors are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

21. a. If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and to $\mathbf{u}_2$ and if $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z}$ must be in $W^\perp$.

b. For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W.

c. The orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{\mathbf{y}}$.

d. If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

e. If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of U.

All vectors are in $\mathbf{R}^n$. Mark each statement True or False. Justify each answer.

22. a. If W is a subspace of $\mathbb{R}^n$ and if $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

b. In the Orthogonal Decomposition Theorem, each term in formula (2) for $\hat{\mathbf{y}}$ is itself an orthogonal projection of $\mathbf{y}$ onto a subspace of W.

c. If $\mathbf{y} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1$ is in a subspace W and $\mathbf{z}_2$ is in $W^\perp$, then $\mathbf{z}_1$ must be the orthogonal projection of $\mathbf{y}$ onto W.

d. The best approximation to $\mathbf{y}$ by elements of a subspace W is given by the vector $\mathbf{y} - \operatorname{proj}_W \mathbf{y}$.

e. If an $n \times p$ matrix U has orthonormal columns, then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

23. Let A be an $m \times n$ matrix. Prove that every vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written in the form $\mathbf{x} = \mathbf{p} + \mathbf{u}$, where $\mathbf{p}$ is in Row A and $\mathbf{u}$ is in Nul A. Also, show that if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then there is a unique $\mathbf{p}$ in Row A such that $A\mathbf{p} = \mathbf{b}$.

24. Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.

a. Explain why $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{v}_1, \dots, \mathbf{v}_q\}$ is an orthogonal set.

b. Explain why the set in part (a) spans $\mathbb{R}^n$.

c. Show that $\dim W + \dim W^\perp = n$.