Math 2331 – Linear Algebra

6.2 Orthogonal Sets

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Orthogonal Sets

A set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) in \( \mathbb{R}^n \) is called an **orthogonal set** if \( \mathbf{u}_i \cdot \mathbf{u}_j = 0 \) whenever \( i \neq j \).

**Example**

Is \( \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) an orthogonal set?

**Solution:** Label the vectors \( \mathbf{u}_1, \mathbf{u}_2, \) and \( \mathbf{u}_3 \) respectively. Then

\[
\mathbf{u}_1 \cdot \mathbf{u}_2 =
\]
\[
\mathbf{u}_1 \cdot \mathbf{u}_3 =
\]
\[
\mathbf{u}_2 \cdot \mathbf{u}_3 =
\]

Therefore, \( \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} \) is an orthogonal set.
Theorem (4)

Suppose \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthogonal set of nonzero vectors in \( \mathbb{R}^n \) and \( W = \text{span}\{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \). Then \( S \) is a linearly independent set and is therefore a basis for \( W \).

Partial Proof: Suppose

\[
\begin{align*}
    c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p &= 0 \\
    (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p) \cdot \mathbf{u}_1 &= 0 \\
    (c_1 \mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2 \mathbf{u}_2) \cdot \mathbf{u}_1 + \cdots + (c_p \mathbf{u}_p) \cdot \mathbf{u}_1 &= 0 \\
    c_1 (\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2 (\mathbf{u}_2 \cdot \mathbf{u}_1) + \cdots + c_p (\mathbf{u}_p \cdot \mathbf{u}_1) &= 0 \\
    c_1 (\mathbf{u}_1 \cdot \mathbf{u}_1) &= 0
\end{align*}
\]

Since \( \mathbf{u}_1 \neq 0 \), \( \mathbf{u}_1 \cdot \mathbf{u}_1 > 0 \) which means \( c_1 = 0 \).

In a similar manner, \( c_2, \ldots, c_p \) can be shown to be all 0. So \( S \) is a linearly independent set.
An **orthogonal basis** for a subspace $W$ of $\mathbb{R}^n$ is a basis for $W$ that is also an orthogonal set.

**Example**

Suppose $S = \{u_1, u_2, \ldots, u_p\}$ is an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$ and suppose $y$ is in $W$. Find $c_1, \ldots, c_p$ so that

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p.$$  

**Solution:**

$$y \cdot u_1 = (c_1 u_1 + c_2 u_2 + \cdots + c_p u_p) \cdot u_1$$

$$y \cdot u_1 = c_1 (u_1 \cdot u_1) + c_2 (u_2 \cdot u_1) + \cdots + c_p (u_p \cdot u_1)$$

$$y \cdot u_1 = c_1 (u_1 \cdot u_1) \quad \Rightarrow \quad c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1}$$

Similarly, $c_2 = \ldots, c_p =$
Orthogonal Basis: Theorem

**Theorem (5)**

Let \( \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\} \) be an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \). Then each \( \mathbf{y} \) in \( W \) has a unique representation as a linear combination of \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \). In fact, if

\[
\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p
\]

then

\[
c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \ldots, p)
\]
Orthogonal Basis: Example

Example

Express \( y = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \) as a linear combination of the orthogonal basis

\[
\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

Solution:

\[
\frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{y \cdot u_3}{u_3 \cdot u_3} =
\]

Hence

\[
y = \_\_\_\_u_1 + \_\_\_\_u_2 + \_\_\_\_u_3
\]
Orthogonal Projections

For a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$, suppose we want to write $\mathbf{y}$ in $\mathbb{R}^n$ as the following:

$$\mathbf{y} = (\text{multiple of } \mathbf{u}) + (\text{multiple a vector } \perp \text{ to } \mathbf{u})$$

$$(\mathbf{y} - \alpha \mathbf{u}) \cdot \mathbf{u} = 0 \implies \mathbf{y} \cdot \mathbf{u} - \alpha (\mathbf{u} \cdot \mathbf{u}) = 0 \implies \alpha =$$

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \quad \text{(orthogonal projection of } \mathbf{y} \text{ onto } \mathbf{u})$$

$$\mathbf{z} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \quad \text{(component of } \mathbf{y} \text{ orthogonal to } \mathbf{u})$$
Example

Let $\mathbf{y} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{0}$ and $\mathbf{u}$.

Solution:

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} =$$

Distance from $\mathbf{y}$ to the line through $\mathbf{0}$ and $\mathbf{u}$ is the distance from $\hat{\mathbf{y}}$ to $\mathbf{y}$:

$$= \| \hat{\mathbf{y}} - \mathbf{y} \| =$$
Orthonormal Sets

A set of vectors \( \{u_1, u_2, \ldots, u_p\} \) in \( \mathbb{R}^n \) is called an **orthonormal set** if it is an orthogonal set of unit vectors.

Orthonormal Basis

If \( W = \text{span}\{u_1, u_2, \ldots, u_p\} \), then \( \{u_1, u_2, \ldots, u_p\} \) is an orthonormal basis for \( W \).

Recall that \( v \) is a unit vector if \( ||v|| = \sqrt{v \cdot v} = \sqrt{v^T v} = 1 \).
### Orthonormal Matrix: Example

**Suppose** $U = [u_1 \ u_2 \ u_3]$ where $\{u_1, u_2, u_3\}$ is an orthonormal set.

$$U^T U = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} [u_1 \ u_2 \ u_3] = \begin{bmatrix} \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$$

### Orthogonal Matrix

It can be shown that

$$UU^T = I.$$

So

$$U^{-1} = U^T$$

(such a matrix is called an **orthogonal matrix**).
### Theorem (6)

An $m \times n$ matrix $U$ has orthonormal columns if and only if $U^T U = I$.

### Theorem (7)

Let $U$ be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

**Proof of part b:** $(U\mathbf{x}) \cdot (U\mathbf{y}) =$