Math 3331  Differential Equations
2.3 Models of Motion

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2.3 Models of Motion

- Motion of a Ball Near Surface of the Earth
  - Without Air Resistance
  - With Air Resistance
    - Linear Model
    - Quadratic Model
Motion of a Ball Near Surface of the Earth

- Gravity Force: \( F_g = -mg \)
  \(- g = 32 \text{ ft/s}^2 = 9.8 \text{ m/s}^2 \)

- Air Resistance: \( F_{air} = R(v) \)
  - Linear Model:
    \[
    R(v) = -kv
    \]
    - \([k] = \text{mass/time}\)
    - valid for small velocities
  - Quadratic Model:
    \[
    R(v) = -k|v|v
    \]
    - \([k] = \text{mass/length}\)
    - valid for larger velocities

- We treat only the linear model.
2.3 Gravity Force and Air Resistance

Solution of the Motion without Air Resistance

\[ v' = -g, \quad x' = v \]

\[ \Rightarrow v(t) = -gt + v_0 \]

\[ x(t) = -gt^2/2 + v_0 t + x_0 \]

\[ \Rightarrow t = (v_0 - v)/g \]

\[ x = (v_0^2 - v^2)/(2g) + x_0 \]

\[ \Rightarrow v^2 = v_0^2 + 2g(x_0 - x) \]

Max Height (if \( v_0 > 0 \)):

\[ v = 0 \Rightarrow t_{max} = v_0/g \]

\[ x_{max} = x_0 + v_0^2/(2g) \]

Ground Hit: \( x = 0 \Rightarrow \)

\[ v_g = -\sqrt{v_0^2 + 2gx_0} \text{ (impact velocity)} \]

\[ t_g = \left( v_0 + \sqrt{v_0^2 + 2gx_0} \right)/g \]
Example 1:

Ascending balloon, velocity $15 \, m/s$. At height $100 \, m$ package is dropped. When does package reach ground?

$$g = 9.8 \, m/s^2$$

Initial Values:

$$x_0 = 100 \, m, \quad v_0 = 15 \, m/s$$

$$\Rightarrow t_{max} = 1.5 \, s$$
$$x_{max} = 111.5 \, m$$
$$v_g = 46.7 \, m/s$$
$$t_g = 6.3 \, s$$
2.3 Gravity Force and Air Resistance

Solution of the Motion with Air Resistance

\[ v' = -g - \left( \frac{k}{m} \right)v \]

In CN 2.2, Example p.5, we showed:

\[ y' = ry + a, \quad y(0) = y_0 \]

\[ \Rightarrow \quad y(t) = (y_0 + a/r)e^{rt} - a/r \]

Here: \( y = v, \ a = -g, \ r = -k/m \)

\[ \Rightarrow \quad v(t) = (v_0 + gm/k)e^{-kt/m} - gm/k \]

Integrate this to find \( x \):

\[ x(t) = \int_0^t v(t')dt' + x_0 \]

\[ = \frac{m}{k}(v_0 + gm/k)(1 - e^{-kt/m}) \]

\[ - (gm/k)t + x_0 \]

Terminal Velocity:

\[ v_{term} \equiv \lim_{t \to \infty} v(t) = -gm/k \]
Example 2: (see text, Example 3.8)

$m = 2\, \text{kg}, \quad k = 4\, \text{kg/m} \quad (g = 9.8\, \text{m/s}^2)$

Initial values: $x_0 = 250\, \text{m}, \quad v_0 = 0$

**Question:**
Time of ground hit? Impact velocity?

**Answer:**
Ground hit $\rightarrow$ equation for $t = t_g$:

\[
0 = g\frac{m}{k}^2(1 - e^{-kt/m}) - \frac{gm}{k}t + x_0 \\
= 2.45(1 - e^{-2t}) - 4.9t + 250 \\
= 252.45 - 2.45e^{-2t} - 4.9t
\]

Equation solver $\rightarrow t_g \approx 51.52\, \text{s}$

Impact velocity:

\[
v_g = v(t_g) = 4.9(e^{-2t_g} - 1) \approx -4.9\, \text{m/s}
\]

**Without air resistance:**

\[
t_g = \sqrt{\frac{2x_0}{g}} \approx 7.14\, \text{s} \\
v_g = -\sqrt{2gx_0} \approx -44.3\, \text{m/s}
\]
Graphs for Example 2

- **With air resistance**
- **No air resistance**

- For velocity $v$ vs. time $t$:
  - Graph with a downward slope indicating a decrease in velocity.
  - Graph without any effect, showing a horizontal line.

- For position $x$ vs. time $t$:
  - Graph with a linear decrease indicating a constant acceleration.
  - Graph without any effect, showing a linear decrease.

These graphs illustrate the impact of air resistance on motion.