Math 3331 Differential Equations

5.4 Using the $\mathcal{L}$-Transform to Solve ODE

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- Basic Idea
- Examples

Basic Idea

Basic Idea: $\left\{ \begin{array}{c} \text{IVP} \\ \text{for } y(t): \\ \text{ODE+IC} \end{array} \right\} \xrightarrow{\mathcal{L}} \left\{ \begin{array}{c} \text{algebraic} \\ \text{equation} \\ \text{for } Y(s) \end{array} \right\} \xrightarrow{\text{solve}} Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t)$

Example 8

Ex. 8: $y'' + y = \cos 2t$
$y(0) = 0,\ y'(0) = 1$

$\mathcal{L}$–transform ODE:

$$\mathcal{L}(y'' + y) = \mathcal{L}\{\cos 2t\}$$

$$\begin{aligned} \mathcal{L}(y'') &= s^2Y - sy(0) - y'(0) \\ &= s^2Y - 1 \\ \mathcal{L}(y) &= Y \end{aligned}$$

$$\Rightarrow \mathcal{L}(y'' + y) = (s^2 + 1)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4}$$

$$\Rightarrow (s^2 + 1)Y - 1 = \frac{s}{s^2 + 4}$$

$$\begin{aligned} \Rightarrow Y(s) &= \frac{1}{s^2 + 1} + \frac{s}{(s^2 + 1)(s^2 + 4)} \\ &= \frac{s^2 + s + 4}{(s^2 + 1)(s^2 + 4)} \end{aligned}$$

$y(t) = \mathcal{L}^{-1}(Y)(t)$. From Ex. 5 $\Rightarrow$

$$y(t) = \frac{1}{3}(\cos t + 3\sin t - \cos 2t)$$

Example 9

Ex. 9: $y'' - 2y' - 3y = 0$

$$y(0) = 1,\ y'(0) = 0$$

$$\begin{aligned} \mathcal{L}(y'') &= s^2 Y - s \\ \mathcal{L}(y') &= sY - 1 \end{aligned}$$

$$\Rightarrow \mathcal{L}(y'' - 2y' - 3y) = (s^2 - 2s - 3)Y - s + 2 = 0$$

$$\Rightarrow \ Y(s) = \frac{s-2}{s^2 - 2s - 3},\ y(t) = \mathcal{L}^{-1}(Y)(t)$$

From Ex. 2: $y(t) = (1/4)(e^{3t} + 3e^{-t})$

Example 10

Ex. 10: $y'' - y = e^t$, $y(0) = y'(0) = 0$

$$\mathcal{L}(y'' - y) = (s^2 - 1)Y, \ \mathcal{L}\{e^t\} = \frac{1}{s-1}$$

$$\Rightarrow (s^2 - 1)Y = 1/(s-1)$$

$$\Rightarrow Y(s) = \frac{1}{(s^2-1)(s-1)} = \frac{1}{(s+1)(s-1)^2}$$

Ex. 6 $\Rightarrow y(t) = (e^{-t} - e^t + 2te^t)/4$

Example 11

Ex. 11: $y'' + 2y' + 2y = \cos 2t,\ y(0) = 0,\ y'(0) = 1$

$$\mathcal{L}(y'' + 2y' + 2y) = (s^2Y - 1) + 2(sY) + 2Y = (s^2 + 2s + 2)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4} \Rightarrow (s^2 + 2s + 2)Y - 1 = \frac{s}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+2s+2} + \frac{s}{(s^2+2s+2)(s^2+4)}$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2+2s+2}\} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2+1}\} = e^{-t}\sin t$$

From Ex. 7: $\mathcal{L}^{-1}\{\frac{s}{(s^2+2s+2)(s^2+4)}\}$

$$= \frac{1}{10}(e^{-t}\cos t - 3e^{-t}\sin t - \cos 2t + 2\sin 2t)$$

$$\Rightarrow y(t) = \frac{1}{10}(e^{-t}\cos t + 7e^{-t}\sin t - \cos 2t + 2\sin 2t)$$