Math 3331  Differential Equations

8.1 Introduction to Systems

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/~jiwenhe/math3331
8.1 Introduction to Systems
Definitions and Examples

- Definitions
  - System of First Order ODEs
  - IVP: Existence and Uniqueness of Solution

- Examples

- Reduction of Higher Order Equations

- Worked out Examples from Exercises:
  - 1, 2, 7
System of First Order ODEs

System of 1st order ODEs:

\[
x'_1 = f_1(t, x_1, \ldots, x_n) \\
\vdots \\
x'_n = f_n(t, x_1, \ldots, x_n)
\]

Vector notation:

\[
x = [x_1, \ldots, x_n]^T \\
f = [f_1, \ldots, f_n]^T \\
x' = [x'_1, \ldots, x'_n]^T
\]

\[
x' = f(t, x) \quad (1)
\]

\[n:\ \text{dimension of system}\]
\[n = 2:\ \text{planar system}\]

- (1) is autonomous if \(f\) does not depend on \(t\)
- (1) is non-autonomous if \(f\) depends on \(t\)
Initial Value Problem:
\[ x' = f(t, x), \quad x(t_0) = x_0 \quad (2) \]

**Thm.:** If $f$ is continuous in a region $R$ and has continuous partial derivatives $\partial f_i/\partial x_j$ in $R$, (2) has a unique solution in $R$. 
Example 1

Ex.: 
\[
\begin{align*}
x'_1 &= -ax_1x_2 \\
x'_2 &= ax_1x_2 - bx_2 \\
x'_3 &= bx_2 \\
\end{align*}
\]

\[
x = [x_1, x_2, x_3]^T \\
f(x) = [-ax_1x_2, ax_1x_2 - bx_2, bx_2]^T \\
x' = f(x) \text{ is 3d autonomous system}
\]
Example 2

Ex.: \[ x' = v \]
\[ v' = -x - 0.2v + 2 \cos t \]

is 2d non-autonomous system
Thm. Any system of higher order ODEs in explicit form can be transformed to a (larger) system of first order ODEs.
Example 3

Ex.: \[ x''' + xx'' = \cos t \] (3)

Set \[ x_1 = x, \ x_2 = x', \ x_3 = x'' \]

\[ \Rightarrow \]
\[ x_1' = x' = x_2 \]
\[ x_2' = x'' = x_3 \]
\[ x_3' = x''' = -xx'' + \cos t \]
\[ = -x_1x_3 + \cos t \]

Hence equivalent system:

\[ x_1' = x_2 \]
\[ x_2' = x_3 \]
\[ x_3' = -x_1x_3 + \cos t \] (4)

Given a solution \( x(t) \) of (3) \( \Rightarrow \)
\[ [x(t), x'(t), x''(t)]^T \] is solution of (4)

Conversely: Given a solution \[ x(t) = [x_1(t), x_2(t), x_3(t)]^T \] of (4) \( \Rightarrow \)
\[ x(t) = x_1(t) \] is a solution of (3)
General Higher Order ODEs:

\[ x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \]

Set \( x_1 = x, x_2 = x', \ldots, x_n = x^{(n-1)} \)

\[ \Rightarrow x_1' = x' = x_2 \]
\[ x_2' = x'' = x_3 \]
\[ \vdots \]
\[ x_{n-1}' = x^{(n-1)} = x_n \]
\[ x_n' = x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \]
\[ = f(t, x_1, x_2, \ldots, x_n) \]

\[ \Rightarrow \text{equivalent system:} \]
\[ x_1' = x_2 \]
\[ x_2' = x_3 \]
\[ \vdots \]
\[ x_{n-1}' = x_n \]
\[ x_n' = f(t, x_1, x_2, \ldots, x_n) \]
Exercise 8.1.1

**Ex. 1:** Is the system autonomous? What is the dimension?

\[
\begin{align*}
  x' &= v \\
  v' &= -x - 0.02v + 2\cos t
\end{align*}
\]

is non-autonomous (\(\cos t\)). Dimension: 2
Exercise 8.1.2

\[ \begin{align*}
\theta' &= \omega \\
\omega' &= -(g/L) \sin \theta + (k/m) \omega \end{align*} \]

is autonomous. Dimension: 2

**Ex. 2:** Same questions as in Ex. 1
Exercise 8.1.7

Ex. 7: Show that given functions are solutions of initial value problem

IVP: \[
\begin{align*}
x' &= -4x + 6y \\
y' &= -3x + 5y
\end{align*}
\] \quad \{ \begin{align*} x(0) &= 0 \\
y(0) &= 1 \end{align*} \}; \quad \text{functions} \quad \{ \begin{align*} x(t) &= 2e^{2t} - 2e^{-t} \\
y(t) &= -e^{-t} + 2e^{2t} \end{align*} \}

\[
x'(t) = 4e^{2t} + 2e^{-t}, \quad -4x(t) + 6y(t) = -4(2e^{2t} - 2e^{-t}) + 6(-e^{-t} + 2e^{2t}) = 4e^{2t} + 2e^{-t}
\]
\[
y'(t) = e^{-t} + 4e^{2t}, \quad -3x(t) + 5y(t) = -3(2e^{2t} - 2e^{-t}) + 5(-e^{-t} + 2e^{2t}) = e^{-t} + 4e^{2t}
\]

IC: \quad x(0) = 0, \quad y(0) = 1,

hence \(x(t), y(t)\) are solutions of IVP