Math 3331  Differential Equations
8.2 Geometric Interpretation of Solutions

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8.2 Geometric Interpretation of Solutions

- Definitions
  - Autonomous system and Phase Space Plot
  - Planar Autonomous System

- Example: Predator-Prey System

- Worked out Examples from Exercises:
  - 1, 3, 17, 22, 23, 24, 25
Autonomous system:
\[ x' = f(x), \quad x = [x_1, \ldots, x_n]^T \]
For any \( t \): \( x(t) \in \mathbb{R}^n \)

\[ \Rightarrow \text{RHS doesn't depend explicitly on } t. \]

- Tangent vectors:
  \[ x'(t) = f(x(t)) \]
- Vector field: \( x \rightarrow f(x) \)

- \( \mathbb{R}^n \): phase space
  \( (n = 2): \text{phase plane} \)
- Trajectory: Curve
  \[ \{x(t) \mid t \in I\} \text{ in } \mathbb{R}^n \]
  \( I \): interval on which \( x(t) \) is defined

- \( x(t) \) solution \( \Rightarrow x(t - t_0) \)
  solution: same trajectory!
- If existence and uniqueness, trajectories don’t intersect
Planar Autonomous System

Planar Autonomous system

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

⇒ RHSs \( f \) and \( g \) don’t depend explicitly on \( t \).

- The \( xy \)-plane is the phase plane
- The solution curve \( t \to (x(t), y(t)) \) is a trajectory (or phase plane plot).

Tangent vector:

\[ (x'(t), y'(t)) = (f(x(t), y(t)), g(x(t), y(t))) \]

Vector field:

\[ (x, y) \to (f(x, y), g(x, y)) \]
8.2 Definitions In-Class Exercises

Predator-Prey System

Example: Lotka-Volterra’s predator-prey equations

\[
\begin{align*}
R' &= (a - bF)R \\
F' &= (-c + dR)F
\end{align*}
\]  \hspace{1cm} (1)

\(a, b, c, d > 0\)

\(R\): number of rabbits

\(F\): number of foxes

Parameters:

\(a = 0.4, b = 0.01\)

\(c = 0.3, d = 0.0005\)  \hspace{1cm} (2)

IC: \(R(0) = 40, F(0) = 20\)
Predator-Prey System (cont.)

Composite Graph

Several trajectories
Example

Ex.: \[\begin{align*}
x' &= y \\
y' &= -x - 0.1y
\end{align*}\]

\[x(0) = 0, \quad y(0) = 2\]
Exercise 8.2.1

Ex. 8.2.1: Plot (i) $x_1(t), x_2(t)$ and (ii) the curve $t \rightarrow (x_1(t), x_2(t))$ for $x(t) = [2e^t - e^{-t}, e^{-t}]^T$, i.e. $x_1(t) = 2e^t - e^{-t}$, $x_2(t) = e^{-t}$

Matlab commands:
```
t=linspace(0,2,100); x1=2*exp(t)-exp(-t); x2=exp(-t);
figure(1), plot(t,x1,’k’,t,x2,’k--’), xlabel(’t’), ylabel(’x_1 and x_2’)
figure(2), plot(x1,x2,’k’), xlabel(’x_1’), ylabel(’x_2’), axis([0 15 0 1])
```
Exercise 8.2.3

**Ex. 8.2.3:** Same as Ex. 8.2.1 for

\[ x(t) = [\cos t, \sin t]^T, \text{ i.e. } x_1(t) = \cos t, x_2(t) = \sin t \]
Exercise 8.2.22

Plot $t \rightarrow (x(t), y(t))$ in the $xy$-plane.

Initially, $x(0) = 0$ and $y(0) = 2$, then $y$ decays as $x$ increases, thereafter both $x$ and $y$ oscillate as they decay toward zero.
Exercise 8.2.23

Plot \( t \rightarrow (x(t), y(t)) \) in the \( xy \)-plane.

Initially, \( x(0) = 0 \) and \( y(0) = 2 \). Shortly thereafter, \( y \) decays as \( x \) increases. Soon, both \( x \) and \( y \) begin a seemingly periodic motion.
Exercise 8.2.24

Plot $t \rightarrow (x(t), y(t))$ in the $xy$-plane.

At first, $x$ oscillates mildly about 1, while $y$ oscillates mildly about zero. This would indicate a turning about $(1, 0)$ in the phase plane. The oscillations grow larger until both $x$ and $y$ shoot off to $-\infty$. One possible solution follows.
Exercise 8.2.25

Plot \( t \to (x(t), y(t)) \) in the \( xy \)-plane.

Initially, \( x(0) = 0 \) and \( y(0) = 2 \). Thereafter, \( x \) increases rapidly, then decays asymptotically in an oscillatory manner to about 5 or 6. Meanwhile, \( y \) decays, eventually oscillating about zero. One possible solution follows.
**Exercise 8.2.17**

**Ex. 8.2.17:** Plot (i) solutions $x(t), y(t)$ of IVP as functions of $t$, (ii) trajectory

IVP: $x' = -6x + 10y, \quad y' = -5x + 4y, \quad x(0) = 5, \quad y(0) = 1$. Use *pplane6*: