Math 3331 Differential Equations
8.3 Qualitative Analysis

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8.3 Qualitative Analysis

- Equilibrium Points
  - Equilibrium Points and Nullclines
  - Examples

- Worked out Examples from Exercises:
  - 1, 2, 7
Equilibrium Points and Nullclines

Ex. : \[
\begin{align*}
R' &= (a - bF')R \\
F' &= (-c + dR)F
\end{align*}
\]

Equilibrium points: \( R' = F' = 0 \)
\[
\Rightarrow \begin{cases} 
(a - bF)R &= 0 \\
(-c + dR)F &= 0 
\end{cases}
\]

Solutions:


Equilibrium points → constant solutions of ODE-system:

\([R(t), F(t)]^T = [c/d, a/b]^T\)

\textbf{R-nullcline: } R' = 0
\[
\Rightarrow R = 0 \text{ and } F = a/b
\]

\textbf{F-nullcline: } F' = 0
\[
\Rightarrow F = 0 \text{ and } R = c/d
\]

Equilibrium points are intersections of nullclines
Example

Ex.: \[ x' = (1 - x - y)x \]
\[ y' = (4 - 2x - 7y)y \]

- **x-nullclines:** \( x = 0, \ x + y = 1 \)
- **y-nullclines:** \( y = 0, \ 2x + 7y = 4 \)

Equilibrium points:
- \((0, 0)\), \((0, 4/7)\), \((1, 0)\), \((3/5, 2/5)\)

several solutions, nullclines, and equilibrium points (using pplane6)
Exercise 8.3.1

Ex. 8.3.1: Plot (i) nullclines and (ii) equilibrium points for

\[
\begin{align*}
    x' &= 0.2x - 0.04xy \\
    y' &= -0.1y + 0.005xy
\end{align*}
\]

Nullclines:

\[
\begin{align*}
    x' &= 0 \quad \Rightarrow \quad x = 0 \text{ and } y = 5 \\
    y' &= 0 \quad \Rightarrow \quad y = 0 \text{ and } x = 20
\end{align*}
\]

Equilibria:

\[
\begin{align*}
    \{ [0, 0]^T, [20, 5]^T \}
\end{align*}
\]

Use pplane6:
Exercise 8.3.2

Ex. 8.3.2: Plot (i) nullclines and (ii) equilibrium points for
\[
\begin{align*}
x' &= 4x - 2x^2 - xy \\
y' &= 4y - xy - 2y^2
\end{align*}
\]
Nullclines:
\[
\begin{align*}
x' = 0 & \Rightarrow x = 0 \text{ and } 2x + y = 4 \\
y' = 0 & \Rightarrow y = 0 \text{ and } x + 2y = 4
\end{align*}
\]
Equilibria:
\[
\begin{align*}
\{ [0, 0]^T, [2, 0]^T, [4/3, 4/3]^T, [0, 2]^T \}
\end{align*}
\]
\text{pplane6:}

[Diagram showing nullclines and equilibrium points]
Ex. 8.3.7: Consider \( \begin{cases} x' = 1 - (y - \sin x) \cos x \\ y' = \cos x - y + \sin x \end{cases} \)

(a) Show that \( x(t) = t, \ y(t) = \sin t \) is a solution:

\[
\begin{align*}
x' &= 1, \\
1 - (\sin t - \sin t) \cos t &= 1 \\
y' &= \cos t, \\
\cos t - \sin t + \sin t &= \cos t
\end{align*}
\]
Ex. 8.3.7: Consider \[
\begin{cases}
x' &= 1 - (y - \sin x) \cos x \\
y' &= \cos x - y + \sin x
\end{cases}
\]

(b) Plot solutions:

\[x' = 1(y - \sin(x))\cos(x), \quad y' = \cos(x) - y + \sin(x)\]
Exercise 8.3.7(c)

Consider the system
\[
\begin{align*}
x' &= 1 - (y - \sin x) \cos x \\
y' &= \cos x - y + \sin x
\end{align*}
\]

(c) Show that $y(t) < \sin x(t)$ for all $t$ if $x(0) = \pi/2$, $y(0) = 0$:

Solution of (a) satisfies $y = \sin x$. Trajectories don’t cross $\Rightarrow y(t) < \sin x(t)$ if $y(0) < \sin x(0)$. 