Math 3331 Differential Equations

9.3 Phase Plane Portraits

Text

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9.3 Phase Plane Portraits

- Classification of 2d Systems
- Distinct Real Eigenvalues
  - Phase Portrait
  - Saddle: $\lambda_1 > 0 > \lambda_2$
  - Nodal Source: $\lambda_1 > \lambda_2 > 0$
  - Nodal Sink: $\lambda_1 < \lambda_2 < 0$
- Complex Eigenvalues
  - Center: $\alpha = 0$
  - Spiral Source: $\alpha > 0$
  - Spiral Sink: $\alpha < 0$
- Borderline Cases
  - Degenerate Node: Borderline Case Spiral/Node
  - Degenerate Nodal Source
  - Degenerate Nodal Sink
  - SaddleNode: Borderline Case Node/Saddle
  - Unstable Saddle-Node
  - Stable Saddle-Node

Classification of 2d Systems

$$\mathbf{x}' = A\mathbf{x},\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$T = a + d,\ D = ad - bc$$

$$p(\lambda) = \lambda^2 - T\lambda + D$$

Case A: $T^2 - 4D > 0$

$\Rightarrow$ real distinct eigenvalues

$\lambda_{1,2} = (T \pm \sqrt{T^2 - 4D})/2$

Case B: $T^2 - 4D < 0$

$\Rightarrow$ complex eigenvalues

$\lambda_{1,2} = \alpha \pm i\beta$

$\alpha = T/2,\ \beta = \sqrt{4D - T^2}/2$

Case C: $T^2 - 4D = 0$

$\Rightarrow$ single eigenvalue

$\lambda = T/2$

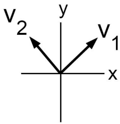
Case A: $T^2 - 4D > 0$

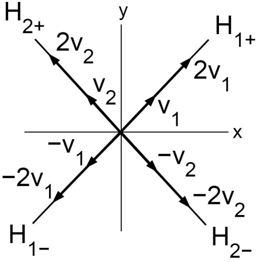
Case A: $T^2 - 4D > 0$

$\Rightarrow$ real distinct eigenvalues

$$\lambda_{1,2} = (T \pm \sqrt{T^2 - 4D})/2$$

- The 4 half line trajectories separate 4 regions of $\mathbf{R}^2$





General Solution:
($\mathbf{v}_1, \mathbf{v}_2$: eigenvectors)

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

$L_{1,2}$: Full lines generated by $\mathbf{v}_{1,2}$

Half line trajectories:

if $c_2 = 0 \Rightarrow \mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1$

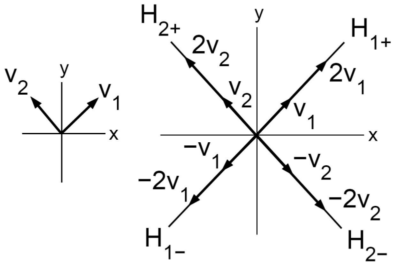
$\Rightarrow$ trajectory is half line

$$H_{1+} = \{\mathbf{x} = \alpha \mathbf{v}_1 \,|\, \alpha > 0\} \text{ if } c_1 > 0$$
$$H_{1-} = \{\mathbf{x} = \alpha \mathbf{v}_1 \,|\, \alpha < 0\} \text{ if } c_1 < 0$$

Same for $H_{2\pm}$ if $c_1 = 0$, $c_2 > 0$ or < 0

Phase Portrait

Phase portrait:
Sketch trajectories. Indicate *direction of motion* by arrows pointing in the direction of increasing t



Direction of Motion on Half Line Trajectories:

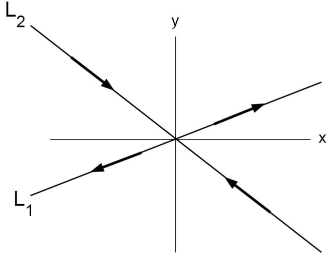
- If $\lambda_1 > 0$ then $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1$
  - moves out to ∞ for $t \to \infty$ (outwards arrow on H_{1+})
  - approaches $\mathbf{0}$ for $t \to -\infty$
- If $\lambda_1 < 0$ then $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1$
  - approaches $\mathbf{0}$ for $t \to \infty$ (inwards arrow on H_{1+})
  - moves out to ∞ for $t \to -\infty$

Saddle: $\lambda_1 > 0 > \lambda_2$

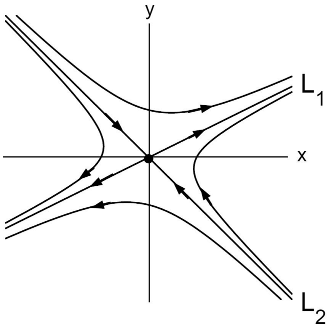
Saddle

$$\lambda_1 > 0 > \lambda_2$$

Half line trajectories



Generic Trajectories



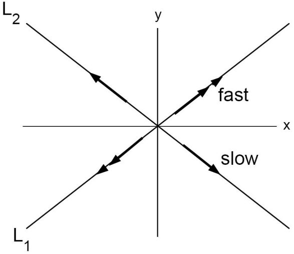
Generic trajectory in each region approaches

- L_1 for $t \to \infty$
- L_2 for $t \to -\infty$

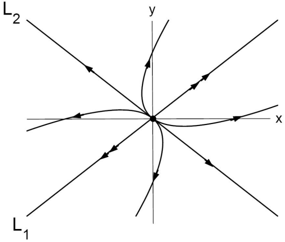
Nodal Source: $\lambda_1 > \lambda_2 > 0$

Nodal source
$\lambda_1 > \lambda_2 > 0$

Half line trajectories



Generic Trajectories



$\longrightarrow\!\!\!\rightarrow$: fast escape to ∞

Generic trajectory is

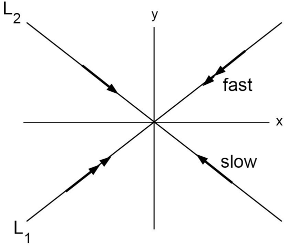
- parallel to L_1 for $t \to \infty$
- tangent to L_2 for $t \to -\infty$

Nodal Sink: $\lambda_1 < \lambda_2 < 0$

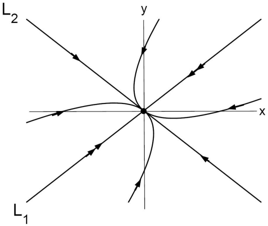
Nodal sink

$\lambda_1 < \lambda_2 < 0$

Half line trajectories



Generic Trajectories



$\longrightarrow\!\!\!\!\rightarrow$: fast approach to $\mathbf{0}$

Generic trajectory is

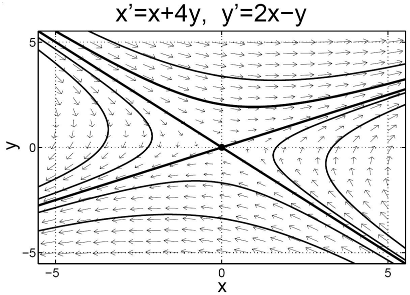
- parallel to L_1 for $t \to -\infty$
- tangent to L_2 for $t \to \infty$

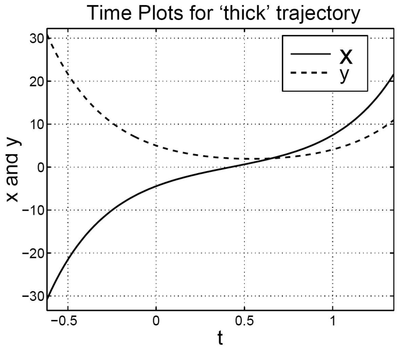
Saddle: Example

Saddle

Ex.: $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$

$\lambda_1 = 3 \quad \leftrightarrow \mathbf{v}_1 = [2, 1]^T$
$\lambda_2 = -3 \leftrightarrow \mathbf{v}_2 = [-1, 1]^T$



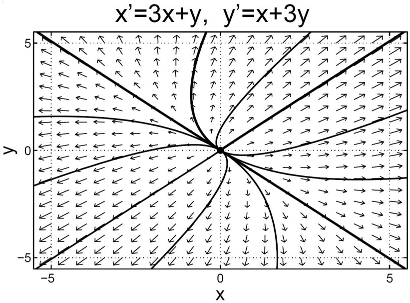


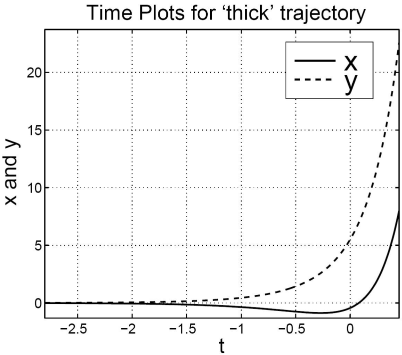
Nodal Source: Example

Nodal Source

Ex.: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$\lambda_1 = 4 \leftrightarrow \mathbf{v}_1 = [1,1]^T$$
$$\lambda_2 = 2 \leftrightarrow \mathbf{v}_2 = [-1,1]^T$$





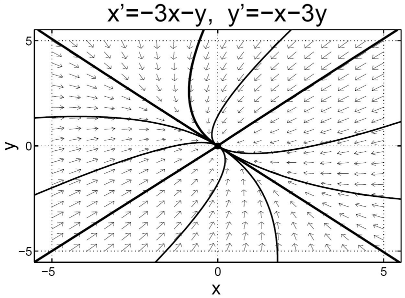
Nodal Sink: Example

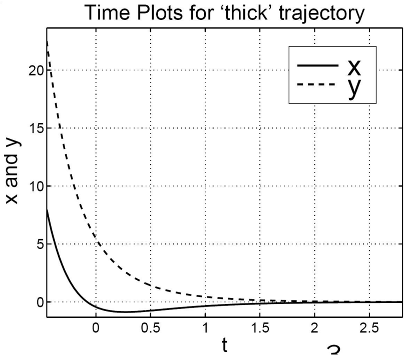
Nodal Sink

Ex.: $A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}$

$\lambda_1 = -4 \leftrightarrow \mathbf{v}_1 = [1, 1]^T$

$\lambda_2 = -2 \leftrightarrow \mathbf{v}_2 = [-1, 1]^T$





Case A: $T^2 - 4D < 0$

Case B: $T^2 - 4D < 0$

$\Rightarrow$ complex eigenvalues

$\lambda_{1,2} = \alpha \pm i\beta$

$\alpha = T/2$, $\beta = \sqrt{4D - T^2}/2$

λ complex

$\Rightarrow$ eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ complex

$\Rightarrow$ no half line solutions

General solution:

$$\mathbf{x}(t) = e^{at}\Big[c_1(\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t) + c_2(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t)\Big]$$

Subcases of Case B

Center: $\alpha = 0$
Spiral Source: $\alpha > 0$
Spiral Sink: $\alpha < 0$

Borderline Case:

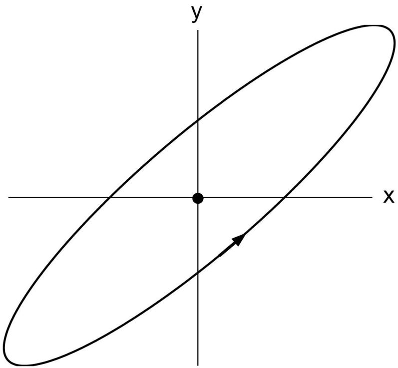
Center ($\alpha = 0$) is border between spiral source ($\alpha > 0$) and spiral sink ($\alpha < 0$).

Center: $\alpha = 0$

Center: $\alpha = 0$

$\Rightarrow$ $\mathbf{x}(t)$ periodic

$\Rightarrow$ trajectories are closed curves



Direction of Rotation: At $\mathbf{x} = [1, 0]^T$, $y' = c$.

If $c > 0$, $\Rightarrow$ counterclockwise,

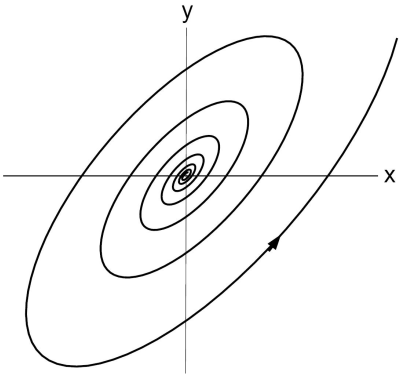
If $c < 0$, $\Rightarrow$ clockwise.

Spiral Source: $\alpha > 0$

Spiral Source: $\alpha > 0$

$\Rightarrow$ growing oscillations

$\Rightarrow$ trajectories are outgoing spirals



Direction of Rotation: At $\mathbf{x} = [1, 0]^T$, $y' = c$.

If $c > 0$, $\Rightarrow$ counterclockwise,

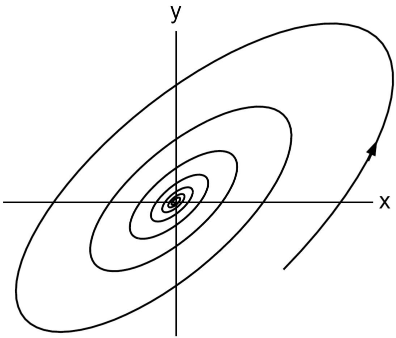
If $c < 0$, $\Rightarrow$ clockwise.

Spiral Sink: $\alpha < 0$

Spiral Sink: $\alpha < 0$

$\Rightarrow$ decaying oscillations

$\Rightarrow$ trajectories are ingoing spirals



Direction of Rotation: At $\mathbf{x} = [1, 0]^T$, $y' = c$.

If $c > 0$, $\Rightarrow$ counterclockwise,

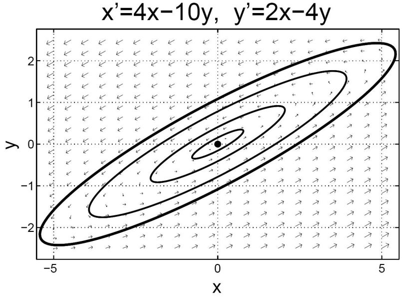
If $c < 0$, $\Rightarrow$ clockwise.

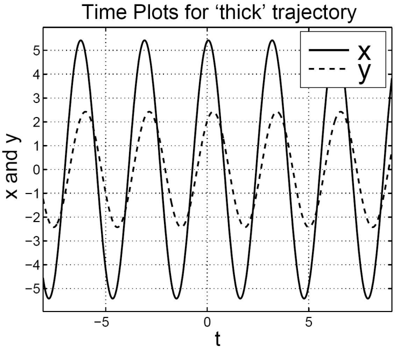
Center: Example

Center

Ex.: $A = \begin{bmatrix} 4 & -10 \\ 2 & -4 \end{bmatrix}$

$$\lambda = 2i \leftrightarrow \mathbf{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$



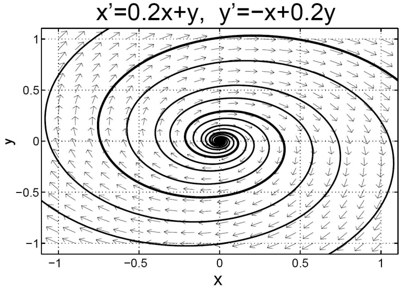


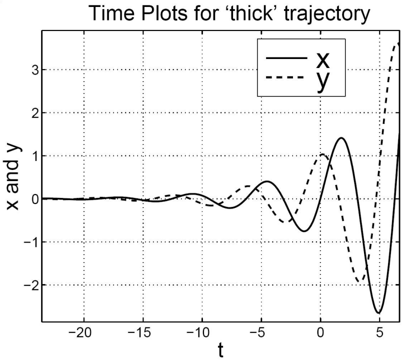
Spiral Source: Example

Spiral Source

Ex.: $A = \begin{bmatrix} 0.2 & 1 \\ -1 & 0.2 \end{bmatrix}$

$$\lambda = 0.2 + i \leftrightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$



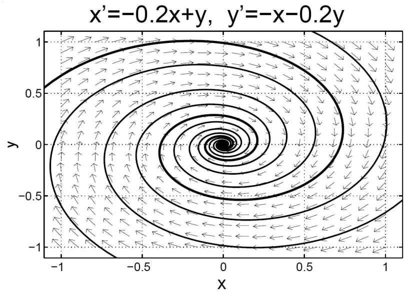


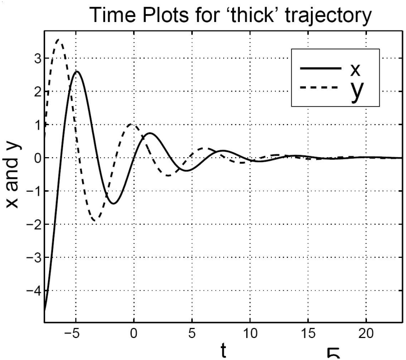
Spiral Sink: Example

Spiral Sink

Ex.: $A = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix}$

$$\lambda = -0.2 + i \leftrightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$





Degenerate Node: Borderline Case Spiral/Node

Degenerate Node: Borderline Case Spiral/Node

- Assume $T^2 - 4D = 0 \Rightarrow$ single eigenvalue $\lambda = T/2$
- Assume generic case: $(A - \lambda I) \neq 0 \Rightarrow$ single eigenvector $\mathbf{v}$
- Let $(A - \lambda I)\mathbf{w} = \mathbf{v} \Rightarrow$ General solution:

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(\mathbf{w} + t\mathbf{v})$$

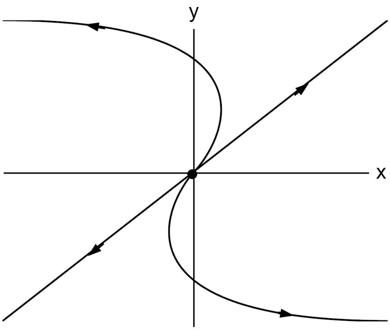
$\Rightarrow$ only two half line solutions on straight line generated by $\mathbf{v}$

Degenerate Nodal Source

Degenerate Nodal Source:

$$T > 0$$

$$\text{borderline case} \begin{cases} \text{nodal source} \\ \text{spiral source} \end{cases}$$

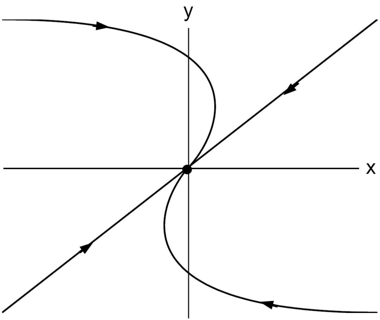


Degenerate Nodal Sink

Degenerate Nodal Sink:

$$T < 0$$

borderline case $\begin{cases} \text{nodal sink} \\ \text{spiral sink} \end{cases}$



SaddleNode: Borderline Case Node/Saddle

SaddleNode: Borderline Case Node/Saddle

- Assume $D = 0$, $T \neq 0 \Rightarrow$ eigenvalues $\lambda_1 = 0$, $\lambda_2 = T$
- Let $\mathbf{v}_1, \mathbf{v}_2$ be the eigenvectors $\Rightarrow$ General solution:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

$\Rightarrow$ – line of equilibrium points generated by $\mathbf{v}_1$

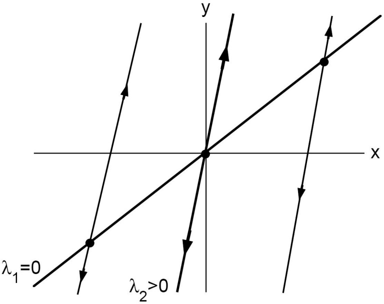
– infinitely many half line solutions on straight lines parallel to line generated by $\mathbf{v}_2$

Unstable Saddle-Node

Unstable Saddle–Node:

$$T > 0$$

$$\text{borderline case} \begin{cases} \text{nodal source} \\ \text{saddle} \end{cases}$$



Stable Saddle-Node

Stable Saddle–Node:

$$T < 0$$

$$\text{borderline case} \begin{cases} \text{nodal sink} \\ \text{saddle} \end{cases}$$