Math 3331  Differential Equations
9.7 Qualitative Analysis of Linear Systems

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9.7 Qualitative Analysis of Linear Systems

- Stability
  - Definitions
  - Examples
  - Theorems
- Stability of 2D Systems
- Worked out Examples from Exercises:
  - 1, 3, 5, 7
Stability: Definitions

\[ x' = Ax, \quad A : n \times n \]  \hspace{1cm} (1)
\[ x(t) = 0 \] is equilibrium solution

Consider general system:
\[ x' = f(x) \] \hspace{1cm} (2)
Assume equilibrium \( x(t) = x_0 \):
\[ f(x_0) = 0 \]

**Def.:**
- \( x_0 \) is stable if for any \( \epsilon > 0 \) there is a \( \delta > 0 \) s.t. \( |x(t) - x_0| < \epsilon \) for all \( t > 0 \) whenever \( |x(0) - x_0| < \delta \).
  (Solutions that start close to \( x_0 \) remain close.)

**Def.:**
- \( x_0 \) is unstable if it is not stable.
  (There are solutions starting arbitrarily close to \( x_0 \) that move ‘far away’ from \( x_0 \).)
- \( x_0 \) is asymptotically stable if \( x_0 \) is stable and there is \( \eta > 0 \) s.t.
  \( x(t) \to x_0 \) for \( t \to \infty \) whenever
  \( |x(0) - x_0| < \eta \).

**Def.:**
- An asymptotically stable equilibrium \( x_0 \) of (2) is a sink.
- An equilibrium \( x_0 \) of (2) is a source if every solution \( x(t) \) with
  \( |x(0) - x_0| \) arbitrarily small eventually moves ‘far away’ from \( x_0 \)
  when \( t \) increases.
9.7 Stability: In-Class Exercises

Stability: Examples

\[ x' = Ax, \quad A : n \times n \]  \hspace{1cm} (1)
\[ x(t) = 0 \] is equilibrium solution

Examples:

Let \( A \) be \( 2 \times 2 \).

The equilibrium \( x_0 = 0 \) of (1) is

- a sink if the phase portrait is a nodal or spiral sink
- a source if the phase portrait is a nodal or spiral source
- unstable if the phase portrait is a saddle
- stable but not asymptotically stable if the phase portrait is a center or stable saddle-node.
Stability: Theorems

Thm.: Let $A$ be $n \times n$

1. If $\text{Re}(\lambda) < 0$ for all eigenvalues of $A$ ($\lambda < 0$ if $\lambda$ is real), then $x(t) \to 0$ for $t \to \infty$ for any solution $x(t)$ of (1). (0 is a sink)

2. If there is an eigenvalue $\lambda$ of $A$ with $\text{Re}(\lambda) > 0$ ($\lambda > 0$ if $\lambda$ is real), then there are solutions $x(t)$ of (1) with $|x(0)|$ arbitrarily small that get arbitrarily large when $t$ increases. (0 is unstable)

3. If $\text{Re}(\lambda) > 0$ for all eigenvalues $\lambda$ of $A$, then every solution $x(t)$ of (1) with $x(0) \neq 0$ gets arbitrarily large when $t$ increases. (0 is a source)

4. If $\text{Re}(\lambda) \leq 0$ for all eigenvalues $\lambda$ of $A$, and for any eigenvalue with $\text{Re}(\lambda) = 0$ every generalized eigenvector is an eigenvector, then 0 is stable. (Ex.: stable saddle-nodes, centers)
Stability of 2D Systems

For $n = 2$:
- $D > 0, \ T < 0 \Rightarrow$ sink
- $D > 0, \ T > 0 \Rightarrow$ source
- $D < 0 \Rightarrow$ saddle $\Rightarrow$ unstable but not source

Ex.: $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} x_0 \\ e^{-t}y_0 \end{bmatrix}$

$\lambda = 0 \leftrightarrow v = [1, 0]^T$:
$0$ is stable

Ex.: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} T & 0 \\ D & 0 \end{bmatrix} \Rightarrow p(\lambda) = \lambda^2$
$\lambda = 0 \leftrightarrow v = [1, 0]^T$
$A^2 = 0 \Rightarrow$ every vector is generalized eigenvector

Solution:
$x(t) = (I + At)x_0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 + tx_0 \end{bmatrix}$

$\Rightarrow$ $0$ is unstable (but not a source)
Exercise 9.7.1

**Ex. 1:** Classify $0$ as unstable equilibrium, stable equilibrium, sink or source of $x' = Ax$ for the given $A$. Verify the classification through a phase portrait.

\[ A = \begin{bmatrix} -0.2 & 2 \\ -2 & -0.2 \end{bmatrix} : \quad D = 4.04 > 0, \quad T = -0.4 < 0 \Rightarrow \text{sink (spiral sink)} \]
Exercise 9.7.3

Ex. 3: Same as Ex. 1 for $A = \begin{bmatrix} -6 & -15 \\ 3 & 6 \end{bmatrix}$

$D = 9$, $T = 0 \Rightarrow$ center $\Rightarrow$ stable (but not sink)

$x' = -6x - 15y$, $y' = 3x + 6y$
Exercise 9.7.5

**Ex. 5:** Same as Ex. 1 for $A = \begin{bmatrix} 0.1 & 2 \\ -2 & 0.1 \end{bmatrix}$.

$D = 4.04, \quad T = 0.2 \Rightarrow$ source (phase portrait: spiral source)

$$x' = 0.1x + 2y, \quad y' = -2x + 0.2y$$
Exercise 9.7.7

**Ex. 7:** Same as Ex. 1 for \( A = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix} \).

\( D = -2 \Rightarrow \text{saddle} \Rightarrow \text{unstable (but not source)} \)