1. Commuting to work requiring getting on a bus near home and then transferring to a second bus. If the waiting time (in minutes) at each stop has a uniform distribution with $A = 0$ and $B = 5$, then it can be shown that the total waiting time $Y$ has the pdf

$$f(y) = \begin{cases} \frac{1}{25}y & 0 \leq y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \leq y \leq 10 \\ 0 & y < 0 \text{ or } y > 10 \end{cases}$$

(a) Sketch a graph of the pdf of $Y$.
(b) Verify that $\int_{-\infty}^{\infty} f(y) dy = 1$.
(c) Compute and sketch the cdf of $Y$.
(d) What is the probability that total waiting time is between 3 and 8 minutes?
(e) Compute $E(Y)$ and $V(Y)$. How do these compare with the expected waiting time and variance for a single bus when the time is uniformly distributed on $[0, 5]$?
(f) Explain how symmetry can be used to obtain $E(Y)$.

Solution:

(a) The graph of the pdf of $Y$ is
(b) We have $f(y) \leq 0$ for any $y$, and

$$\int_{-\infty}^{\infty} f(y)dy = \int_{0}^{5} \frac{1}{25}ydy + \int_{5}^{10} \left(\frac{2}{5} - \frac{1}{25}y\right) dy = \left[\frac{1}{50}y^2\right]_0^5 + \left[\frac{2}{5}y - \frac{1}{50}y^2\right]_5^{10}$$

$$= \frac{1}{2} + ((4 - 2) - (2 - \frac{1}{2})) = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) For $0 \leq y < 5$,

$$F(y) = \int_{-\infty}^{y} f(u)du = \int_{0}^{y} \frac{1}{25}udu = \left[\frac{1}{50}u^2\right]_0^y = \frac{1}{50}y^2,$$

for $5 \leq y < 10$,

$$F(y) = \int_{-\infty}^{y} f(u)du = \int_{0}^{5} \frac{1}{25}udu + \int_{5}^{y} \left(\frac{2}{5} - \frac{1}{25}u\right) du = \left[\frac{1}{50}u^2\right]_0^5 + \left[\frac{2}{5}u - \frac{1}{50}u^2\right]_5^y$$

$$= \frac{1}{2} + \left(\frac{2}{5}y - \frac{1}{50}y^2\right) - (2 - \frac{1}{2}) = \frac{2}{5}y - \frac{1}{50}y^2 - 1.$$

Then the cdf of $Y$ is

$$F(y) = \begin{cases} 
0 & y < 0 \\
\frac{1}{50}y^2 & 0 \leq y < 5 \\
\frac{2}{5}y - \frac{1}{50}y^2 - 1 & 5 \leq y \leq 10 \\
1 & y > 10
\end{cases}$$

And the graph of the cdf of $Y$ is

(d) The probability that total waiting time is between 3 and 8 minutes is

$$P(3 \leq Y \leq 8) = F(8) - F(3) = \left(\frac{2}{5}8 - \frac{1}{50}8^2 - 1\right) - \left(\frac{1}{50}3^2\right) = \frac{46}{50} - \frac{9}{50} = \frac{37}{50} = 0.74.$$
(e) The expected waiting time and variance of \( Y \) are

\[
E(Y) = \int_{-\infty}^{\infty} y f(y) dy = \int_{0}^{5} \frac{1}{25} y^2 dy + \int_{5}^{10} \left(\frac{2}{5} y - \frac{1}{25} y^2\right) dy = \left[ \frac{1}{75} y^3 \right]_{0}^{5} + \left[ \frac{2}{15} y^2 - \frac{1}{75} y^3 \right]_{5}^{10} \\
= \frac{5}{3} + ((20 - \frac{40}{3}) - (5 - \frac{5}{3})) = 5.
\]

\[
E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y) dy = \int_{0}^{5} \frac{1}{25} y^3 dy + \int_{5}^{10} \left(\frac{2}{5} y^2 - \frac{1}{25} y^3\right) dy = \left[ \frac{1}{100} y^4 \right]_{0}^{5} + \left[ \frac{2}{15} y^3 - \frac{1}{100} y^4 \right]_{5}^{10} \\
= \frac{25}{4} + ((\frac{400}{3} - 100) - (\frac{50}{3} - \frac{25}{4})) = \frac{350}{12} \\
\]

\[
V(Y) = E(Y^2) - [E(Y)]^2 = \frac{350}{12} - 5^2 = \frac{50}{12} \approx 4.1667 
\]

Let \( X \) be the waiting time and variance for a single bus. Then the pdf of \( X \) is

\[
f_X(x) = \begin{cases} 
\frac{1}{5} & 0 \leq x \leq 5 \\
0 & x < 0 \text{ or } x > 5
\end{cases}
\]

Then the expected waiting time and variance of \( X \) are

\[
E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{0}^{5} \frac{1}{5} x dx = \left[ \frac{1}{10} x^2 \right]_{0}^{5} = \frac{5}{2} = 2.5
\]

\[
E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{0}^{5} \frac{1}{5} x^2 dx = \left[ \frac{1}{10} x^3 \right]_{0}^{5} = \frac{25}{3}
\]

\[
V(X) = E(X^2) - [E(X)]^2 = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{25}{3} - \frac{25}{12} \approx 2.0833.
\]

Thus \( E(Y) = 2E(X) \) and \( V(Y) = V(X) \).

(f) Since the pdf of \( Y \) is symmetric about 5, obviously \( E(Y) = 5 \).
2. A result called **Chebyshev’s inequality** states that for any probability distribution of an rv $X$ and any number $k$ that is at least 1, $P(|X - \mu| \geq k\sigma) \leq 1/k^2$. In words, the probability that the value of $X$ lies at least $k$ standard deviations from its mean is at most $1/k^2$.

(a) What is the value of the upper bound of Chebyshev’s inequality for $k = 1, 2, 3$?

(b) Obtain this probability in the case of $X$ having a normal distribution for $k = 1, 2, 3$, and compare to the upper bound of Chebyshev’s inequality. Appendix Table A.3 of Textbook contains the values $\Phi(-3) = 0.0013$, $\Phi(-2) = 0.0228$, $\Phi(-1) = 0.1587$, $\Phi(1) = 0.8413$, $\Phi(2) = 0.9772$, $\Phi(3) = 0.9987$.

**Solution:**

(a) The value of the upper bound of Chebyshev’s inequality $1/k^2$ for $k = 1, 2, 3$ are

$$1/1^2 = 1, \quad 1/2^2 = 1/4 = 0.25, \quad 1/3^2 = 1/9 \approx 0.11$$

(b) In the case of $X$ having a normal distribution with mean $\mu$ and standard deviation $\sigma$, we define $Z = \frac{X-\mu}{\sigma}$. Then, $Z$ has the standard normal distribution, and we obtain the probability

$$P(|X - \mu| \geq k\sigma) = 1 - P(|X - \mu| \leq k\sigma) = 1 - P(-k\sigma \leq X - \mu \leq k\sigma)$$

$$= 1 - P(-k \leq \frac{X-\mu}{\sigma} \leq k) = 1 - P(-k \leq Z \leq k) = 1 - (P(Z \leq k) - P(Z \leq -k))$$

$$= 1 - (\Phi(k) - \Phi(-k)) = \begin{cases} 1 - (\Phi(1) - \Phi(-1)) = 0.3174 & \text{for } k = 1 \\ 1 - (\Phi(2) - \Phi(-2)) = 0.0456 & \text{for } k = 2 \\ 1 - (\Phi(3) - \Phi(-3)) = 0.0026 & \text{for } k = 3 \end{cases}$$

These values are considerably less than the bounds 1, 0.25, and 0.11 given by Chebyshev’s inequality.
3. Let \( X \) have a normal distribution with mean \( \mu \) and standard deviation \( \sigma \).

   (a) Find the pdf of \( Y = e^X \). The distribution of \( Y \) is lognormal.

   (b) Find the pdf of \( Z = X^2 \). The distribution of \( Z \) is chi-squared with 1 degree of freedom for \( \mu = 0 \) and \( \sigma = 1 \).

**Solution:**

(a) The rv \( X \) has a normal distribution with pdf

\[
 f_X(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x - \mu)^2}{2\sigma^2}}
\]

From the relation \( Y = e^X \), we have \( y > 0 \) and \( y = e^x \), which is increasing and has the inverse \( x = \ln y \). Then the pdf of \( Y \) is

\[
 f_Y(y) = f_X(\ln y) \frac{d}{dy}(\ln y) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} \frac{1}{y}
\]

Thus the distribution of \( Y \) is lognormal.

(b) Note that the function \( z = x^2 \) is not one-to-one. The cdf of \( Z \) is given by

\[
 F_Z(z) = P(Z \leq z) = P(X^2 \leq z) = P(-\sqrt{z} \leq X \leq \sqrt{z}) = F_X(\sqrt{z}) - F_X(-\sqrt{z}).
\]

Thus \( Z \) has the pdf:

\[
 f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \left( F_X(\sqrt{z}) - F_X(-\sqrt{z}) \right) = \frac{1}{2\sqrt{z}} \left( f_X(\sqrt{z}) + f_X(-\sqrt{z}) \right).
\]

The pdf of \( X \) is given by

\[
 f_X(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x - \mu)^2}{2\sigma^2}}
\]

Then we have

\[
 f_Z(z) = \frac{1}{\sqrt{2\pi \sigma}} \frac{1}{2\sqrt{z}} \left( e^{-\frac{(\sqrt{z} - \mu)^2}{2\sigma^2}} + e^{-\frac{(\sqrt{z} + \mu)^2}{2\sigma^2}} \right)
\]

For \( \mu = 0 \) and \( \sigma = 1 \), \( Z \) has the pdf

\[
 f_Z(z) = \frac{1}{\sqrt{2\pi \sqrt{z}}} e^{-\frac{z}{2}}
\]

thus has the chi-squared distribution with 1 degree of freedom

---

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.