HW12 Solutions
Math 3338-10853: Probability (Fall 2006), Dr. Jiwen He

1. 
   a. \( P(X = 1, Y = 1) = p(1,1) = .20 \)
   
   b. \( P(X \leq 1 \text{ and } Y \leq 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .42 \)
   
   c. At least one hose is in use at both islands. \( P(X \neq 0 \text{ and } Y \neq 0) = p(1,1) + p(1,2) + p(2,1) + p(2,2) = .70 \)
   
   d. By summing row probabilities, \( p_i(x) = .16, .34, .50 \text{ for } x = 0, 1, 2, \text{ and by summing} \) 
   column probabilities, \( p_j(y) = .24, .38, .38 \text{ for } y = 0, 1, 2. \) \( P(X \leq 1) = p_0(0) + p_0(1) = .50 \)
   
   e. \( P(0,0) = .10, \text{ but } p_0(0) \cdot p_0(0) = (16)(.24) = .0384 \neq .10, \text{ so } X \text{ and } Y \text{ are not independent.} \)

7. 
   a. \( p(1,1) = .030 \)
   
   b. \( P(X \leq 1 \text{ and } Y \leq 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .120 \)
   
   c. \( P(X = 1) = p(1,0) + p(1,1) + p(1,2) = .100; P(Y = 1) = p(0,1) + \ldots + p(5,1) = .300 \)
   
   d. \( P(\text{overflow}) = P(X + 3Y > 5) = 1 - P(X + 3Y \leq 5) = 1 - P[(X,Y) = (0,0) \text{ or } \ldots \text{ or } (5,0) \text{ or} \) 
   \( (0,1) \text{ or } (1,1) \text{ or } (2,1)] = 1 -.620 = .380 \)
   
   e. The marginal probabilities for \( X \) (row sums from the joint probability table) are \( p_i(0) = .05, p_i(1) = .10, p_i(2) = .25, p_i(3) = .30, p_i(4) = .20, p_i(5) = .10; \) those for \( Y \) (column sums) are \( p_j(0) = .5, p_j(1) = .3, p_j(2) = .2. \) It is now easily verified that for every \( (x,y), \) 
   \( p(x,y) = p(x) \cdot p(y), \) so \( X \) and \( Y \) are independent.

11. 
   a. \( p(x,y) = \frac{e^{-\lambda} \lambda^x}{x!} \cdot \frac{e^{-\mu} \mu^y}{y!} \text{ for } x = 0, 1, 2, \ldots; y = 0, 1, 2, \ldots \)
   
   b. \( p(0,0) + p(0,1) + p(1,0) = e^{-\lambda-\mu} \left[ 1 + \lambda + \mu \right] \)
   
   c. \( P(X+Y = m) = \sum_{k=0}^{m} P(X = k, Y = m-k) = \sum_{k=0}^{m} \frac{e^{-\lambda-\mu} \lambda^k \mu^{m-k}}{k! (m-k)!} \)
   
   \[ \frac{e^{-(\lambda+\mu)} \sum_{k=0}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) \lambda^k \mu^{m-k}}{m!} = \frac{e^{-(\lambda+\mu)}(\lambda + \mu)^m}{m!} \] 
   so the total \# of errors \( X+Y \) also has a Poisson distribution with parameter \( \lambda + \mu. \)