HW6 Solutions

Math 3338-10853: Probability (Fall 2006), Dr. Jiwen He

28.

a. $E(X) = \sum_{x=0}^{4} x \cdot p(x)$

$= (0)(.08) + (1)(.15) + (2)(.45) + (3)(.27) + (4)(.05) = 2.06$

b. $V(X) = \sum_{x=0}^{4} (x - 2.06)^2 \cdot p(x) = (0 - 2.06)^2(.08) + \ldots + (4 - 2.06)^2(.05)$

$= .339488 + .168540 + .001620 + .238572 + .188180 = .9364$

c. $\sigma_x = \sqrt{.9364} = .9677$

d. $V(X) = \left[\sum_{x=0}^{4} x^2 \cdot p(x)\right] - (2.06)^2 = 5.1800 - 4.2436 = .9364$

31.

a. $E(X) = (13.5)(.2) + (15.9)(.5) + (19.1)(.3) = 16.38$,
$E(X^2) = (13.5)^2(.2) + (15.9)^2(.5) + (19.1)^2(.3) = 272.298$,
$V(X) = 272.298 - (16.38)^2 = 3.9936$

b. $E(25X - 8.5) = 25\,E(X) - 8.5 = (25)(16.38) - 8.5 = 401$

c. $V(25X - 8.5) = V(25X) = (25)^2 V(X) = (625)(3.9936) = 2496$

d. $E[h(X)] = E[X - .01X^2] = E(X) - .01E(X^2) = 16.38 - 2.72 = 13.66$

33. $E(X) = \sum_{x=1}^{\infty} x \cdot p(x) = \sum_{x=1}^{\infty} x \cdot \frac{c}{x^3} = c\sum_{x=1}^{\infty} \frac{1}{x^2}$, but it is a well-known result from the theory of infinite series that $\sum_{x=1}^{\infty} \frac{1}{x^2} < \infty$, so E(X) is finite.

40. $V(aX + b) = \sum_x [aX + b - E(aX + b)]^2 \cdot p(x) = \sum_x [aX + b - (a\mu + b)]^2 p(x)$

$$= \sum_x [aX - (a\mu)]^2 p(x) = a^2 \sum_x [X - \mu]^2 p(x) = a^2 V(X).$$

45. $M_X(t) = \Sigma\, e^{xt} p(x) = \Sigma\, e^{xt}(.5)^x = \Sigma\, (.5e^t)^x = \frac{.5e^t}{1 - .5e^t}$ or $\frac{e^t}{2 - e^t}$, since the sum ranges from $x=1$ to $x=\infty$. From this, $E(X) = M'_X(0) = \left.\frac{2e^t}{(2 - e^t)^2}\right|_{t=0} = 2$. Next, $E(X^2) = M''_X(0) = \left.\frac{2e^t(2 + e^t)}{(2 - e^t)^3}\right|_{t=0} = 6$, from which $V(X) = 6 - 2^2 = 2$.

47. Following Example 3.29, $M_X(t) = \Sigma\, e^{xt}p(x) = \Sigma\, e^{xt}pq^{x-1} = pe^t\, \Sigma\, (e^tq)^{x-1} = pe^t\dfrac{1}{1-e^tq}$ =

$\dfrac{pe^t}{1-(1-p)e^t}$. The MGF of Y exactly corresponds to this format with $p = .75$ (and $q = .25$). Hence, Y is a geometric random variable with parameter $p = .75$, and the PMF of Y is $p_Y(y) = .75(.25)^{y-1}$ for $y = 1, 2, 3, \ldots$.

54. $R_X(t) = \ln[M_X(t)] \rightarrow R'_X(t) = \dfrac{M'_X(t)}{M_X(t)} \rightarrow R'_X(0) = \dfrac{M'_X(0)}{M_X(0)} = \dfrac{\mu}{1} = \mu$. Next, using the quotient rule, $R''_X(t) = \dfrac{M_X(t)M''_X(t) - M'_X(t)M'_X(t)}{[M_X(t)]^2} \rightarrow R''_X(0) = \dfrac{M_X(0)M''_X(0) - M'_X(0)^2}{[M_X(0)]^2}$ =

$\dfrac{1 \cdot E(X^2) - \mu^2}{[1]^2} = E(X^2) - \mu^2 = \sigma^2$.