HW9 Solutions

Math 3338-10853: Probability (Fall 2006), Dr. Jiwen He

1.

a. P(X ≤ 1) = $\int_{-\infty}^{1} f(x)dx = \int_0^1 \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_0^1 = .25$

b. P(.5 ≤ X ≤ 1.5) = $\int_{.5}^{1.5} \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_{.5}^{1.5} = .5$

c. P(X > 1.5) = $\int_{1.5}^{\infty} f(x)dx = \int_{1.5}^{2} \frac{1}{2}xdx = \frac{1}{4}x^2\Big]_{1.5}^{2} = \frac{7}{16} \approx .438$

4.

a. $\int_{-\infty}^{\infty} f(x;\theta)dx = \int_0^{\infty} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx = -e^{-x^2/2\theta^2}\Big]_0^{\infty} = 0-(-1) = 1$

b. P(X ≤ 200) = $\int_{-\infty}^{200} f(x;\theta)dx = \int_0^{200} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx$

$= -e^{-x^2/2\theta^2}\Big]_0^{200} \approx -.1353 + 1 = .8647$

P(X < 200) = P(X ≤ 200) ≈ .8647, since x is continuous.

P(X ≥ 200) = 1 - P(X < 200) ≈ .1353

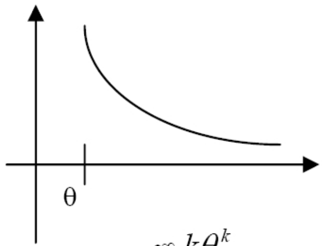
c. P(100 ≤ X ≤ 200) = $\int_{100}^{200} f(x;\theta)dx = -e^{-x^2/20{,}000}\Big]_{100}^{200} \approx .4712$

d. For x > 0, P(X ≤ x) =

$\int_{-\infty}^{x} f(y;\theta)dy = \int_0^{x} \frac{y}{e^2} e^{-y^2/2\theta^2} dx = -e^{-y^2/2\theta^2}\Big]_0^{x} = 1 - e^{-x^2/2\theta^2}$

10.

a.



b. $= \int_{-\infty}^{\infty} f(x;k,\theta)dx = \int_{\theta}^{\infty} \frac{k\theta^k}{x^{k+1}}dx = \theta^k \cdot \left(-\frac{1}{x^k}\right)\Big]_{\theta}^{\infty} = \frac{\theta^k}{\theta^k} = 1$

c. P(X ≤ b) = $\int_{\theta}^{b} \frac{k\theta^k}{x^{k+1}}dx = \theta^k \cdot \left(-\frac{1}{x^k}\right)\Big]_{\theta}^{b} = 1 - \left(\frac{\theta}{b}\right)^k$

d. P(a ≤ X ≤ b) = $\int_{a}^{b} \frac{k\theta^k}{x^{k+1}}dx = \theta^k \cdot \left(-\frac{1}{x^k}\right)\Big]_{a}^{b} = \left(\frac{\theta}{a}\right)^k - \left(\frac{\theta}{b}\right)^k$

13.

a. $1 = \int_1^{\infty} \frac{k}{x^4}dx \Rightarrow 1 = \frac{-k}{3}x^{-3}\Big|_1^{\infty} \Rightarrow 1 = 0 - (-\frac{k}{3})(1) \Rightarrow 1 = \frac{k}{3} \Rightarrow k = 3$

b. cdf: F(x)= $\int_{-\infty}^{x} f(y)dy = \int_1^x 3y^{-4}dy = -\frac{3}{3}y^{-3}\Big|_1^x = -x^{-3} + 1 = 1 - \frac{1}{x^3}$. So

$$F(x) = \begin{cases} 0, & x \le 1 \\ 1 - x^{-3}, & x > 1 \end{cases}$$

c. P(x > 2) = 1 – F(2) = 1 – $\left(1 - \frac{1}{8}\right) = \frac{1}{8}$ or .125;

$P(2 < x < 3) = F(3) - F(2) = \left(1 - \frac{1}{27}\right) - \left(1 - \frac{1}{8}\right) = .963 - .875 = .088$

18.

a. E(X) = $\int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^2 x \cdot \frac{1}{2}xdx = \frac{1}{2}\int_0^2 x^2 dx = \left.\frac{x^3}{6}\right]_0^2 = \frac{8}{6} \approx 1.333$

b. E(X^2) = $\int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2 \frac{1}{2}xdx = \frac{1}{2}\int_0^2 x^3 dx = \left.\frac{x^4}{8}\right]_0^2 = 2,$

So Var(X) = E(X^2) – [E(X)]^2 = $2 - \left(\frac{8}{6}\right)^2 = \frac{8}{36} \approx .222$, $\sigma_X \approx .471$

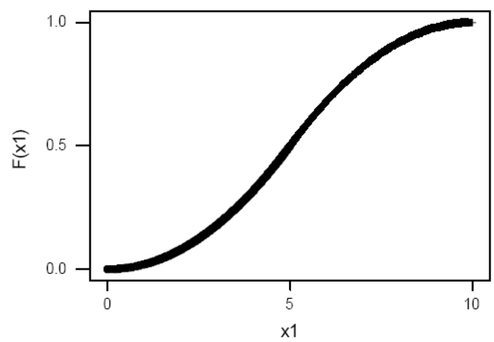
c. From **g** , E(X^2) = 2

24.

a. For $0 \le y \le 5$, F(y) = $\int_0^y \frac{1}{25}udu = \frac{y^2}{50}$

For $5 \le y \le 10$, F(y) = $\int_0^y f(u)du = \int_0^5 f(u)du + \int_5^y f(u)du$

$$= \frac{1}{2} + \int_0^y \left(\frac{2}{5} - \frac{u}{25}\right)du = \frac{2}{5}y - \frac{y^2}{50} - 1$$



b. For $0 < p \le .5$, p = F(y_p) = $\frac{y_p^2}{50} \Rightarrow y_p = (50p)^{1/2}$

For $.5 < p \le 1$, p = $\frac{2}{5}y_p - \frac{y_p^2}{50} - 1 \Rightarrow y_p = 10 - 5\sqrt{2(1-p)}$

c. E(Y) = 5 by straightforward integration, and similarly V(Y)= $\frac{50}{12}$ = 4.1667 . For the waiting time X for a single bus, E(X) = 2.5 and V(X) = $\frac{25}{12}$.

d. Since the PDF of *Y* is symmetric about 5, obviously *E*(*Y*) = 5.

28.

a. E(X) = $\int_{\theta}^{\infty} x \cdot \frac{k\theta^k}{x^{k+1}} dx = k\theta^k \int_{\theta}^{\infty} \frac{1}{x^k} dx = \left.\frac{k\theta^k x^{-k+1}}{-k+1}\right]_{\theta}^{\infty} = \frac{k\theta}{k-1}$

b. E(X) = ∞

c. E(X^2) = $k\theta^k \int_{\theta}^{\infty} \frac{1}{x^{k-1}} dx = \frac{k\theta^2}{k-2}$, so

Var(X) = $\left(\frac{k\theta^2}{k-2}\right) - \left(\frac{k\theta}{k-1}\right)^2 = \frac{k\theta^2}{(k-2)(k-1)^2}$

d. Var(x) = ∞, since E(X^2) = ∞.

e. E(X^n) = $k\theta^k \int_{\theta}^{\infty} x^{n-(k+1)} dx$, which will be finite if n – (k+1) < -1, i.e. if n<k.