2.5 Independence
Definition of independent events

- **Definition:** Two events $A$ and $B$ are independent if
  
  
  $$P(A|B) = P(A)$$

  and are dependent otherwise.
  
  - The equality in the definition implies the following equality (and vice versa)
    
    $$P(B|A) = P(B)$$

  - It is also straightforward to show that if $A$ and $B$ are independent, then so are the following pairs of events: (1) $A'$ and $B$, (2) $A$ and $B'$, and (3) $A'$ and $B'$.

- **2.31: Tossing a die** $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$, and $C = \{1, 2, 3, 4\}$. We have

  $$P(A) = \frac{1}{2}, \quad P(A|B) = \frac{1}{3}, \quad P(A|C) = \frac{1}{2}.$$ 

  That is, $A$ and $B$ are dependent, whereas $A$ and $C$ are independent.

- **2.32** Let $A$ and $B$ be mutually exclusive with $P(A) > 0$. Since $A \cap B = \emptyset$, then $P(A|B) = 0 \neq P(A)$, so $A$ and $B$ can not be independent. For example, $A = \{\text{carisblue}\}$ and $B = \{\text{carisred}\}$, $A$ and $B$ are mutually exclusive, then dependent.
When $A$ and $B$ are Independent

- **Proposition:** $A$ and $B$ are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

- **Proof:**

$$P(A \cap B) = P(A|B) \cdot P(B) = P(A) \cdot P(B)$$

where the second equality is valid if and only if $A$ and $B$ are independent.

- **Definition of the independence of more than two events:** Events $A_1, \ldots, A_n$ are *mutually independent* if for every $k$ ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, \ldots, i_k$,

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k})$$

- **2.35:** Let $A_i$ denote the event that the lifetime of cell $i$ exceeds $t_0$ ($i = 1, \ldots, 6$). We assume that $A_i$’s are independent events and that $P(A_i) = .9$ for every $i$ since the cells are identical.

$$P(\text{system lifetime exceeds } t_0) = P[(A_1 \cap A_2 \cap A_3) \cup (A_4 \cap A_5 \cap A_6)]$$

$$= P(A_1 \cap A_2 \cap A_3) + P(A_4 \cap A_5 \cap A_6)$$

$$- P[(A_1 \cap A_2 \cap A_3) \cap (A_4 \cap A_5 \cap A_6)]$$

$$= (.9)(.9)(.9) + (.9)(.9)(.9) - (.9)(.9)(.9)(.9)(.9)(.9) = .927$$