**Expectation**

+ **Definition.** Let $X$ be a discrete RV with set of possible values $\Omega$ and pmf $p(x)$. The expectation of $X$, denoted by $E(X)$, is:

$$E(X) = \sum_{x \in \Omega} x \cdot p(x)$$

+ **Remark.** $E(X)$ is also known as the expected value of $X$, or the mean of $X$, or the first moment of $X$.

+ **We assume that the summation converges absolutely.** i.e.,

$$\sum_{x \in \Omega} |x| \cdot p(x) < \infty$$

+ **Examples.** Trivial RV. Let $X$ be constant, i.e.,

$$p(x) \begin{cases} 1 & x = \omega \\ 0 & x \neq \omega \end{cases}$$

Then

$$p(\omega) = \sum_{\omega \in \Omega} 1 \quad \omega = \omega \quad \text{and}$$

$$E(X) = \omega$$
- Bernoulli r.v./indicator r.v. Let \( X \) be any r.v. that can take only the values 0 or 1, then \( X \) is said to be a Bernoulli r.v./indicator r.v. If we define the event on which \( X = 1 \),

\[ A = \{ \omega : X(\omega) = 1 \} \]

then \( X \) is said to be the indicator of \( A \).

Then 

\[ P(\omega) = \begin{cases} 1 - p, & X = 0 \\ p, & X = 1 \end{cases} \]

and 

\[ E(X) = 0 \quad (= 1 \cdot p + 0 \cdot (1 - p)) \]

- Uniform r.v. Let \( X \) be uniform on the integers \( \{1, 2, \ldots, n\} \). Then

\[ P(j) = \frac{1}{n}, \quad 1 \leq j \leq n \]

and 

\[ E(X) = \frac{1}{n} \sum_{j=1}^{n} j = \frac{1}{2} (n+1) \quad \text{the average of \( P_j \)} \]

- Triangular r.v. In this case, we have 

\[ P(j) = 1, \quad P(c) = n^2 \min(n+c, n-j) \]

\[ = \sum_{j=0}^{n-2} (n-j), \quad 0 \leq j \leq n \]

and 

\[ E(X) = 0 \quad (= \sum_{j=0}^{n-2} (n-j) + \sum_{j=n}^{n} j n^2 (n+j)) \]
Example. Let $X$ be the sum of the scores of two fair dice. Then

$$
\phi(j) = 6 - 2 \min\{j-1, 13-j\}, \quad 2 \leq j \leq 12
$$

and

$$
E(\bar{X}) = 7
$$

Geometric mean. Let $\bar{X}$ be a geometric RV with parameter $p$. Then

$$
E(\bar{X}) = \sum_{j=1}^{\infty} j \phi(1-p)^{j-1} = p \sum_{j=1}^{\infty} \left[- \frac{d}{dp} (1-p)^j \right] = q^{-1}
$$

Tail sum. When $\bar{X} \geq 1$ and $\bar{X}$ is integer valued, show that

$$
E(\bar{X}) = \sum_{j=0}^{\infty} \left\{1 - \phi(j) \right\}
$$

Proof.

$$
E(\bar{X}) = \sum_{j=1}^{\infty} j \phi(j) = \phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(5) + \cdots
$$

$$
= \sum_{j=1}^{\infty} \phi(j) + \sum_{j=2}^{\infty} \phi(j) \cdot \frac{1}{2^j} + \frac{\phi(3)}{2^3} + \cdots
$$

$$
= [1 - F(1)] + [1 - F(2)] + [1 - F(3)] + \cdots = \frac{1}{2} \sum_{j=1}^{\infty} 2^{-j} F(1)\]
Expectation of functions

Theorem: Let $X$ and $Y$ be discrete, with

$Y = g(X)$. 

Then

$E(Y) = \sum_{x \in D} g(x) \varphi(x)$. 

Proof: For some fixed $y$,

$\sum_{x \in D} g(x) \varphi(x) = \sum_{x \in D} Y \varphi(x) = \sum_{x \in D} y \varphi(x)$

$\sum_{x \in D} g(x) = y$

Given the distribution $\varphi(x)$, $Y$ has distribution of $X$,

$\varphi(Y) = \sum_{x \in D} \varphi(x)$

$g(x) = y$

Then

$E(Y) = \sum_{y} Y \varphi(Y) = \sum_{y} \left[ \sum_{x \in D} \varphi(x) \right]$ 

$= \sum_{y} \left[ \sum_{x \in D} g(x) \varphi(x) \right]$

$= \sum_{x \in D} g(x) \varphi(x)$

Remark: We do not need to find the distribution of $Y$ in order to find its mean.

2. It is not true in general that

$E[g(X)] = g[E(X)]$. 
Corollary: Linear transformation

\[ E[a\bar{X} + b] = a\ E[X] + b \]

Proof:

\[ E[a\bar{X} + b] = \sum_x (a \cdot x + b) \cdot p(x) = a \sum_x x \cdot p(x) + b \sum_x p(x) \]

\[ = a \ E[X] + b \]

Corollary: Linearity of expectation

\[ E[g(\bar{X}) + h(\bar{X})] = E[g(\bar{X})] + E[h(\bar{X})] \]

Proof:

\[ E[g(\bar{X}) + h(\bar{X})] = \sum_x (g(x) + h(x)) \cdot p(x) \]

\[ = \sum_x g(x) \cdot p(x) + \sum_x h(x) \cdot p(x) \]

\[ = E[g(\bar{X})] + E[h(\bar{X})]. \]

Example: Let \( \bar{Y} \) be the sum of the scores of two fair dice. Then \( \bar{Y} = \bar{X} + n + 1 \) where \( \bar{X} \) is the triangular \( n \) and \( n = 6 \).

\[ E[\bar{Y}] = E[\bar{X}] + 6 + 1 = 0 + 6 + 1 = 7. \]