Chapter Two: Probability (I)
2.1 Sample Spaces and Events
The sample space of an experiment

- **Experiment**: any action or process whose outcome is subject to uncertainty.
- **Sample space of an experiment**: denoted by Ω, the set of all possible outcomes of that experiment.
- **Sample point**: denoted by ω, a point in Ω.
  - The sample space provides a model of an ideal experiment in the sense that, by definition, every thinkable outcome of the experiment is completely described by one, and only one, sample point.
    \[ \forall \text{outcome of the experiment}, \exists! \omega \in \Omega \Rightarrow \text{outcome} = \omega \]
- **Examples:**
  - **2.1**: For an experiment consisting of examining a single fuse to see whether it is defective, the sample space \( \Omega_1 = \{N, D\} \), where \( N \) represents not defective, \( D \) represents defective.
  - **2.2**: For an experiment consisting of examining three fuses in sequence, the sample space \( \Omega_3 = \Omega_1 \times \Omega_1 \times \Omega_1 = \{(\omega_1, \omega_2, \omega_3), \omega_i \in \{N, D\}\} \).
  - **2.4**: For an experiment consisting of testing each battery as it comes off an assembly line until we first observe a success, \( \Omega = \{S, FS, FFS, FFFS, \ldots\} \), which contains an infinite number of possible outcomes.
Events

- **Event**: denoted by capital letters, is a subset of \( \Omega \).
  - It is meaningful to talk about an event \( A \) only when it is clear for every outcome of the experiment whether the event \( A \) has or has not occurred.
    \[
    \forall \omega \in \Omega, \text{ we have } \omega \in A \text{ or } \omega \notin A.
    \]

- **Elementary event**: also called **simple event**, is an event consisting of exactly one outcome:
  elementary event = sample point.
  An event \( A \) occurs if and only if one of the elementary event \( \omega \) in \( A \) occurs.

- **Certain event**: \( \Omega \), which always occurs regardless the outcome of the experiment.

- **Impossible event**: \( \emptyset \).

- **Example 2.7**:
  - For \( \Omega \) defined in example 2.4, compound events include
    \[
    A = \{ S, FS, FFS \} = \text{ the event that at most three batteries are examined}
    \]
    \[
    E = \{ FS, FFFS, FFFFFS, \ldots \} = \text{ the event that an even number of batteries are examined.}
    \]
Operations with events

- **Complement:** The complement of an event $A$ is denoted by $A^c$ or $A'$, and is the event that $A$ does not occur.
  \[ A^c = \{ \omega | \omega \notin A \}; \quad \Omega^c = \emptyset; \quad \emptyset^c = \Omega; \quad (A^c)^c = A. \]

- **Union:** The union $A \cup B$ of two events $A$ and $B$ is the event consisting of the occurrence of at least one of the events $A$ and $B$.
  \[ A \cup B = \{ \omega | \omega \in A \text{ or } \omega \in B \}. \]

- **Intersection:** The intersection $A \cap B$ of two events $A$ and $B$ is the event consisting of the occurrence of both events $A$ and $B$.
  \[ A \cap B = \{ \omega | \omega \in A \text{ and } \omega \in B \}. \]

- **Mutually exclusive events:** Two events are said mutually exclusive if $A \cap B = \emptyset$, i.e., the event $A \cap B$ is impossible.

- **Figure 2.1 - Venn diagrams:** An event is nothing but a set, so *elementary set theory* can be used to study events.
Relations among events

- **A ⊂ B**: The occurrence of **A** implies the occurrence of **B**, i.e., if \( \omega \in A \), then \( \omega \in B \).

- **A ⊃ B**: The occurrence of **A** is implied by the occurrence of **B**, i.e., if \( \omega \in B \), then \( \omega \in A \).

- **A = B**: The events **A** and **B** are identical.
  
  \[ A = B \text{ if and only if } A \subset B \text{ and } B \subset A. \]

- **A \cap B^c**: **A** but not **B** occurs.

- **A – B**: If **A \supset B**, we denote \( A \cap B^c \) by **A – B**.

- **A + B**: If **A \cap B = \emptyset**, we denote \( A \cup B \) by **A + B**.

- **A \cap B = \emptyset**: means that **A \subset B^c** and **B \subset A^c**, i.e., if **A** and **B** are mutually exclusive, the occurrence of **A** implies the non-occurrence of **B** and vice versa.

- **A – A \cap B**: the occurrence of **A** but not of both **A** and **B**. Thus **A – A \cap B = A \cap B^c**.

- **Indicator**: The indicator of an event **A** is \( I_A : \Omega \rightarrow \{1, 0\} \) s.t.

  \[ I_A(\omega) = \begin{cases} 
  1 & \text{if } \omega \in A, \\
  0 & \text{if } \omega \notin A.
  \end{cases} \]

  We have then

  \[ A = B \text{ if and only if } I_A = I_B \]

  \[ I_{A \cap B} = I_A \cdot I_B \]

  \[ I_{A \cup B} = I_A + I_B - I_A \cdot I_B \]
Various Laws

- **Commutative law:**
  
  \[
  A \cup B = B \cup A \\
  A \cap B = B \cap A
  \]

- **Associative law:**
  
  \[
  (A \cup B) \cup C = A \cup (B \cup C) =: A \cup B \cup C \\
  (A \cap B) \cap C = A \cap (B \cap C) =: A \cap B \cap C
  \]

- **Distributive law:**
  
  \[
  (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\
  (A \cap B) \cup C = (A \cup C) \cap (B \cup C)
  \]

- **De Morgan’s law:**
  
  \[
  (A \cup B)^c = A^c \cap B^c \\
  (A \cap B)^c = A^c \cup B^c
  \]

- **Proofs:** Based on the relation

  \[
  \text{Left } \equiv \text{ Right} \quad \text{if and only if} \quad \left\{ \begin{array}{l}
  (1) \quad \text{Left } \subset \text{ Right} \\
  (2) \quad \text{Right } \subset \text{ Left}
  \end{array} \right.
  \]