Final Exam
Introduction to PDE
MATH 3363-25820 (Fall 2009)

Name and UH-ID:

This exam has 4 questions, for a total of 40 points. Please answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Upon finishing PLEASE write and sign your pledge below: *On my honor I have neither given nor received any aid on this exam.* ____

1 Rules

Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given in Section 2, without proof, on any question in the exam.

2 Given

You may use the following without proof:

The eigen-solution to

$$X'' + \lambda X = 0, \quad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

Orthogonality condition for sines: for any $L > 0$,

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2, & m = n \\ 0 & m \neq n \end{cases}$$

A useful result derived from the Divergence Theorem,

$$\iint_D v\Delta v dV = -\iint_D |\nabla v|^2 dV + \int_{\partial D} v\nabla v \cdot n dS$$

for and 2D or 3D region D with closed boundary ∂D.

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3 Questions

$\boxed{\text{10 points}}$ 1. A bar with initial temperature profile $f(x) > 0$, with ends held at 0^oC, will cool as $t \to \infty$; and approach a steady-state temperature 0^oC. However, whether or not all parts of the bar start cooling initially depends on the shape of the initial temperature profile. The following heat problem may enable you to discover the relationship:

(a) Solve the heat problem on the interval $0 \leq x \leq 1$,

$$u_t = u_{xx}, \quad u(0,t) = 0 = u(1,t), \quad u(x,0) = f(x),$$

where

$$f(x) = -\frac{1}{2}\sin 3\pi x + \frac{3}{2}\sin \pi x.$$

(b) Show that for some x, $0 < x < 1$, $u_t(x,0)$ is positive (i.e., warming) and for others it is negative (i.e, cooling).
Hint: in Figure 1, $u(x,t_0)$ is plotted for $t_0 = 0, 0.2, 0.5, 1$. You only need to find some x for which u_t is positive/negative.

(c) How is the sign of $u_t(x,0)$ (i.e., warming/cooling) related to the shape of the initial temperature profile? How is the sign of $u_t(x,t)$, $t > 0$ (i.e., warming/cooling), related to subsequent temperature profiles?

Hint: the PDE gives $u_t = u_{xx}$ and the sign of u_{xx} gives the concavity of $u(x,t)$.

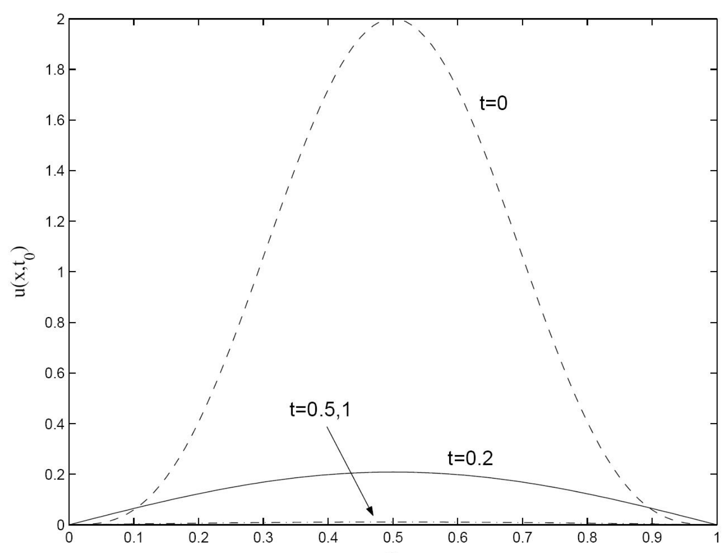

Figure 1. Plots of $u(x,t_0)$ for $t_0 = 0, 0.2, 0.5, 1$

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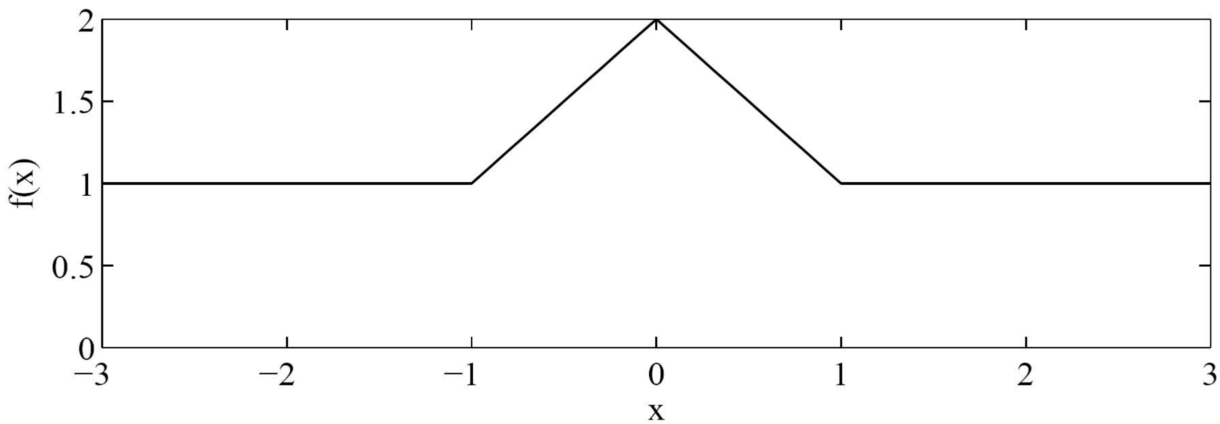

Figure 2. Plot of $f(x)$

10 points 2. Consider the quasi-linear PDE

$$\frac{\partial u}{\partial t} + (1-u)\frac{\partial u}{\partial x} = 0, \quad u(x,0) = f(x)$$

where the initial condition $f(x)$ (shown in Fig. 2) is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \le x \le 0 \\ 2 - x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(a) Find the parametric solution using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x).

(b) At what time t_{sh} and location x_{sh} does your parametric solution break down?

(c) For each of $s = -1, 0, 1$, write down the characteristic x as a functions of t and plot the characteristic in the space-time plane (xt).

(d) The tables below are useful as a plotting aid. Fill in the tables using your result from (a) to obtain u and x at the s-values listed at time $t = 1/2, 1$.

$$
\begin{array}{lll}
 & s = & -1 \quad 0 \quad 1 \\
t = \frac{1}{2} & u = & \\
 & x = &
\end{array}
$$

$$
\begin{array}{lll}
 & s = & -1 \quad 0 \quad 1 \\
t = 1 & u = & \\
 & x = &
\end{array}
$$

(e) Illustrate the time evolution of the solution by sketching the ux-profiles $u(x,t)$ for $t = 0$, 1/2, 1.

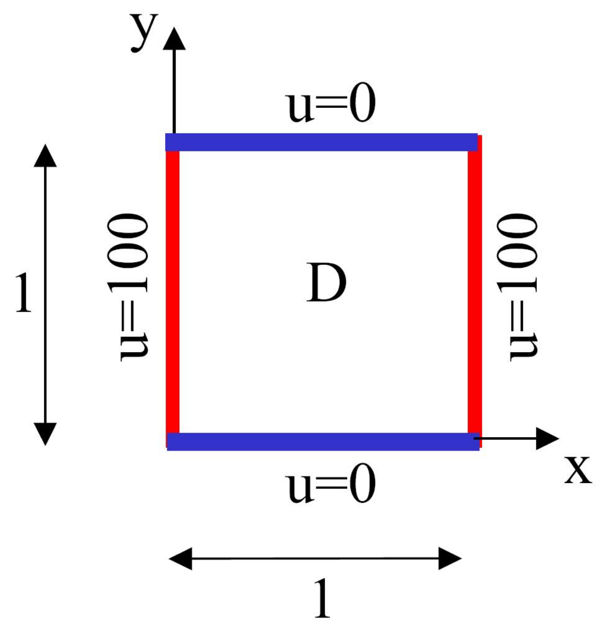

Figure 3. Unit square D and Inhomogeneous BCs

$\boxed{\text{10 points}}$ 3. (a) Solve the Heat Problem on the 2D unit square $D = \{(x, y) : 0 \le x, y \le 1\}$

$$u_t = \Delta u, \quad (x, y) \in D, \quad t > 0,$$

subject to inhomogeneous BCs

$$u(x, y, t) = \begin{cases} 100, & x = 0 \text{ or } 1, \quad 0 < y < 1, \\ 0, & \text{otherwise on } \partial D \end{cases}$$

and initial condition

$$u(x, y, 0) = 0, \quad (x, y) \in D.$$

Hint: first derive the steady-state (equilibrium) solution u_E, set $v = u - u_E$, then transform the given heat problem for u into the homogeneous heat problem for v and solve it for v.

(b) Prove the solution to (a) is unique.
Hint: you'll need to use a result derived from the Divergence Theorem (on the given page).

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$\boxed{\text{10 points}}$ 4. Consider the Boundary Value Problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial D \end{aligned}$$

where D is a simply-connected 2d region with piecewise smooth boundary ∂D.

(a.1) State the Maximum Principle for u on D.

(a.2) If $f = 25$ at each point on the boundary ∂D, what is u in D? Explain your answer.

(b.1) Now let D be the disc of radius R centered at the origin,

$$D = \{(x, y) : x^2 + y^2 \leq R^2\}.$$

Name and state (without proof) another property of u which gives the value of u at the center of the disc in terms of the values of u on the boundary

$$\partial D = \{(x, y) : x^2 + y^2 = R^2\}.$$

(b.2) Use this result to find $u(0, 0)$ if on the boundary, u takes the values

$$u(R, \theta) = \begin{cases} 25, & -\pi/2 \leq \theta \leq \pi/2, \\ 26, & \pi/2 \leq \theta \leq \pi, \\ 24, & \pi \leq \theta \leq 3\pi/2. \end{cases}$$

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1.

1 (cont.)

2.

2 (cont.)

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3.

3 (cont.)

3 (cont.)

4.