Preparation for the Final

Basic Set of Problems that you should be able to do:

- all problems on your tests (1- 3 and their samples)

- extra practice problems in this documents.

The final will be a mix of problems like ones on Basic Set of Problems.

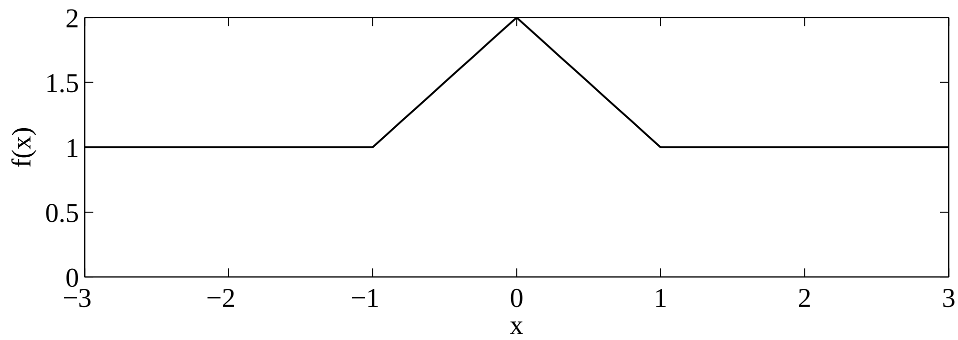

Figure 1: Plot of $f(x)$.

Part A

Consider the following quasi-linear PDE,

$$\frac{\partial u}{\partial t} + (1+u)\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = f(x)$$

where the initial condition is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \le x \le 0 \\ 2 - x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

1. [3 points] Sketch $f(x)$ vs. x. **Solution:** See Figure 1.

2. [6 points] Find the parametric solution. First write down the relevant ODEs for dx/dr, dt/dr, du/dr. Please take the initial conditions $t = 0$ and $x = s$ at $r = 0$. What is the initial condition (i.e. at $r = 0$) for u? Solve for t, u and x (in that order) as functions of r, s.

Solution: We can write the PDE as

$$(1, 1+u, 0) \cdot \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1\right) = 0$$

Thus the parametric solution is defined by the ODEs

$$\frac{dt}{dr} = 1, \qquad \frac{dx}{dr} = 1 + u, \qquad \frac{du}{dr} = 0$$

with initial conditions at $r = 0$,

$$t = 0, \qquad x = s, \qquad u = u(x,0) = u(s,0) = f(s)$$

Integrating the ODEs and imposing the ICs gives

$$\begin{aligned} t &= r \\ u &= f(s) \\ x &= (1+f(s))\,r+s = (1+f(s))\,t+s \end{aligned}$$

3. [5 points] At what time t_s and position x_s does a shock first form? Hint: you might need to consider negative values of $f'(s)$.

Solution: The Jacobian is

$$J = \frac{\partial(x,t)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r} = 0 - (f'(s)\,r+1) = -(f'(s)\,t+1)$$

Shocks occur (the solution breaks down) where $J=0$, i.e. where

$$t = -\frac{1}{f'(s)}$$

The first shock occurs at

$$t_s = \min\left(-\frac{1}{f'(s)}\right) = -\frac{1}{\min f'(s)} = 1$$

where $\min f'(s) = -1$. Since $f'(1/2) = -1$, the $s = 1/2$ characteristic can be used to find the shock location at $t = t_s = 1$,

$$x_s = \left(1 + f\left(\frac{1}{2}\right)\right)1 + \frac{1}{2} = \left(1 + \left(2 - \frac{1}{2}\right)\right)1 + \frac{1}{2} = 3.$$

4. [6 points] Write down x in terms of t, s and $f(s)$. For each of $s = -1, 0, 1$, write down x as a function of t and plot it in the xt-plane up to the shock time $t = t_s$ you found in 3. Put all three curves in the same plot. Label where the shock occurs. You have just plotted the three important characteristics.

Solution: Note that the $s = -1, 0, 1$ characteristics are given by

$$\begin{aligned} s &= -1 : x = (1+f(-1))\,t - 1 = 2t-1 \\ s &= 0 : x = (1+f(0))\,t + 0 = 3t \\ s &= 1 : x = (1+f(1))\,t + 1 = 2t+1 \end{aligned}$$

These are plotted in Figure 2.

5. [4.5 points] Fill in the tables below:

$t = \frac{1}{2}$

$s=$	-1	0	1
$u=$	1	2	1
$x=$	0	$\frac{3}{2}$	2

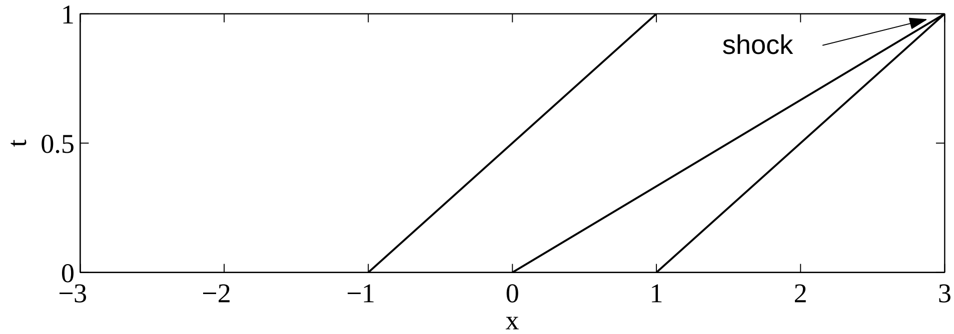

Figure 2: Plot of characteristics for question 4.

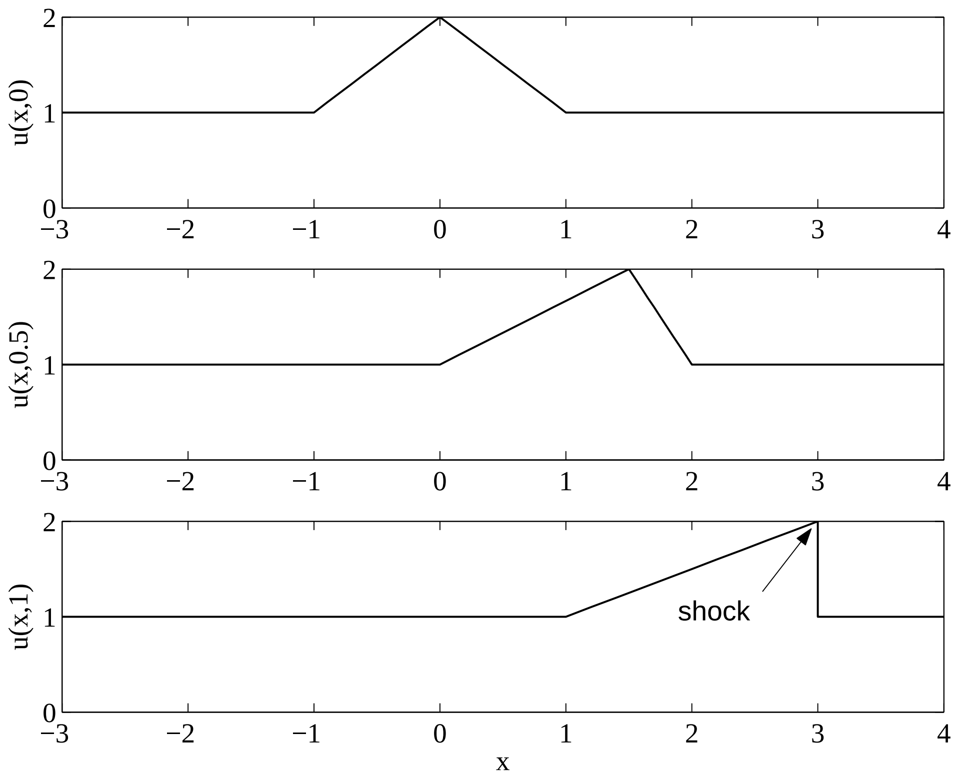

Figure 3: Plot of $u(x, t_0)$ for part A for $t_0 = 0$, 0.5 and 1.

$t = t_s = 1$

$s =$	-1	0	1
$u =$	1	2	1
$x =$	1	3	3

5. [3 points] Plot the three points (x, u) from the table for $t = 1/2$, in the xu-plane and connect the dots. You have just plotted $u(x, t_0)$ at $t_0 = 1/2$.

Solution: See the middle plot in Figure 3.

6. [3 points] Plot the three points (x, u) in the table for $t = t_s$, in the xu-plane and connect the dots. Label the shock. You have just plotted $u(x, t_s)$.

Solution: See the bottom plot in Figure 3.

Part B

[12 marks]

Consider the Boundary Value Problem

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad D \\ u &= f \quad \text{on} \quad \partial D \end{aligned}$$

where D is a simply-connected 2D region with piecewise smooth boundary ∂D.

(i) State the Maximum Principle for u on D. If $f = 10$ at each point on the boundary ∂D, what is u in D? Explain your answer.

(ii) Now let D be the disc of radius R centered at the origin,

$$D = \left\{ (x, y) : x^2 + y^2 \leq R^2 \right\}.$$

Name and state (without proof) another property of u which gives the value of u at the center of the disc in terms of the values of u on the boundary $\partial D = \{(x, y) : x^2 + y^2 = R^2\}$. Use this result to find $u(0, 0)$ if on the boundary, u takes the values

$$u\,(R, \theta) = \begin{cases} 90, & -\pi/2 \leq \theta \leq \pi/2, \\ 25, & \pi/2 \leq \theta \leq \pi, \\ 7, & \pi \leq \theta \leq 3\pi/2. \end{cases}$$

Solution: (i) Maximum principle: solution to laplace's equation takes min/max on boundary. Thus $u = 10$.

(ii)

$$\begin{aligned} u\,(0, 0) &= \frac{1}{2\pi} \int_0^{2\pi} f\,(\theta)\, d\theta \\ &= \frac{1}{2\pi} \left(90\pi + 25\frac{\pi}{2} + 7\frac{\pi}{2} \right) \\ &= \frac{1}{2} \left(90 + 25\frac{1}{2} + 7\frac{1}{2} \right) \\ &= \frac{123}{2} \end{aligned}$$

Part C

[8 marks]

Find the smallest eigenvalue of the Sturm-Liouville problem

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D \\ v &= 0 \quad \text{on} \quad \partial D \end{aligned}$$

where D is a quarter-disc $D = \{(r, \theta) : 0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi/2\}$. You may assume that the smallest eigenvalue is associated with the eigen-function that satisfies the above Sturm-Liouville problem and is non-zero on the interior of D. Explain your answer. Hint: use the given info on page 2. You do not need to solve the PDE.

Solution: the eigen-function that is zero on ∂D and nonzero on the interior of D is

$$v = J_2\left(r j_{2,1}\right) \sin 2\theta$$

and the smallest eigenvalue is

$$\lambda = \lambda_{21} = j_{2,1}^2$$

Part D

[20 marks]

We consider the steady-state temperature on a simply-connected 2D domain D with piecewise-smooth boundary ∂D,

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad D \\ u &= f \quad \text{on} \quad \partial D \end{aligned}$$

NOTE: you do not need to solve the PDE to answer any of the questions below!

(i) Suppose D is a square of side length 1 and u takes the values as shown in Figure 1. Let the origin $(x, y) = (0, 0)$ be the lower left-hand corner of the square.

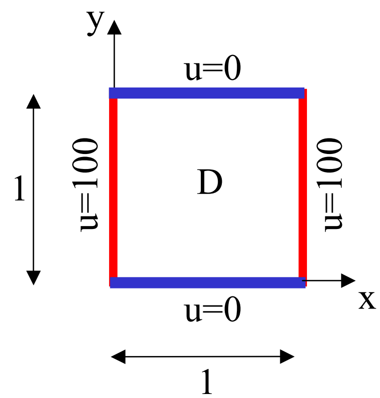

Figure 1:

(a) List the lines of symmetry.

Solution: $x = 1/2$, $y = 1/2$.

(b) Give a symmetry argument to find the steady-state temperature u at the center $(x, y) = (1/2, 1/2)$ and at $(3/4, 3/4)$.

Solution: rotate about diagonal $y = x$, add, get (by max principle) $u = 100$. Then both temps are 90.

(c) What is temperature gradient ∇u (proportional to heat flux) at the center of the square?

Solution: zero, since lines of symmetry cross, i.e. heat flow lines. On vertical line of sym, $u_x = 0$, on horiz, $u_y = 0$. Thus where lines cross, $\nabla u = 0$.

(d) What is the direction (up, down, right, left or zero) of the heat flux (i.e. gradient ∇u) at the points $(x, y) = (1/2, 3/4)$, $(1/2, 1/4)$, $(1/4, 1/2)$?

Solution: up, down, right

(ii) Suppose D is a square of side length $1/2$ whose boundary is either kept at a specific temperature or is insulated, as shown in Figure 2. Assume the origin $(x, y) = (0, 0)$ is the lower left-hand corner of the square.

(a) Use a symmetry argument and your answer in (i) to find the steady-state temperature at the points $(x, y) = (1/2, 0)$ and $(1/4, 1/4)$.

Solution: same as upper half of square. $u\,(1/2, 0)$ is same as in center of square in (i)(b), 45. And also at $(1/4, 1/4)$.

(b) Draw the level curves.

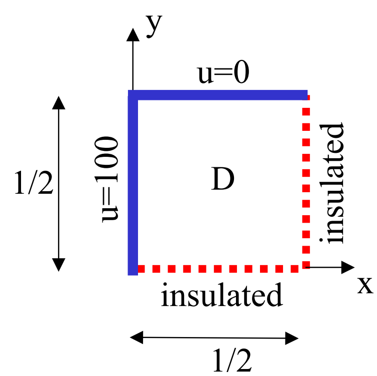

Figure 2:

(iii) Let D be the half disc of radius 1, insulated along the straight side and kept at 90^o and 0 along the curved boundaries in the upper and lower right hand planes, respectively, as shown in Figure 3.

(a) Use a symmetry argument to find the temperature at $(x, y) = (0, 0)$ (point A) and $(x, y) = \left(\frac{1}{2}, 0\right)$ (point B).

Solution: Same as left half of disc, boundary kept at 90 upper plane, 0 lower plane. Symmetry, rotate about x-axis, then entire boundary held at 90, so $u = 90$ everywhere. But center line did not move, so $u = 45$ along center line. Points A and B are on center line, so temp is 45 on each.

(b) Sketch the level (isothermal) curves (using solid lines) and the heat flow lines (using dashed lines).

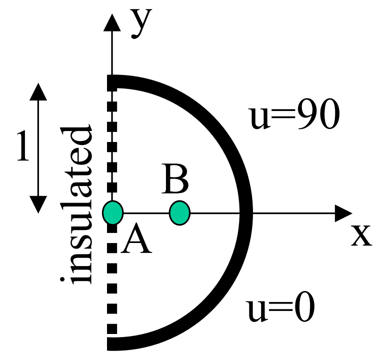

Figure 3:

Part E

1. [8 marks] Find an eigenvalue λ and corresponding eigenfunction v for the right triangle

$$D = \left\{(x, y) : 0 < y < \sqrt{2}x, \quad 0 < x < 1\right\}$$

with side lengths 1 and $\sqrt{2}$. v and λ satisfy the Sturm-Liouville Problem

$$\begin{aligned} \nabla^2 v + \lambda v &= 0 \quad \text{in} \quad D, \\ v &= 0 \quad \text{on} \quad \partial D. \end{aligned}$$

Hint: you may use the eigenfunctions derived in-class for the rectangle, without derivation. You may find constructing a table useful for $2m^2 + n^2$ $(n, m = 1, 2, 3)$

2. [5 BONUS marks] Find a function that is zero on the boundary of the triangle, nonzero and smooth on the interior, and use it to obtain an upper bound on the smallest eigenvalue of the triangle in (b). You don't have to evaluate the integrals; just set them up.

Solution: Eigenfunctions on rectangle are

$$v_{mn} = \sin(m\pi x)\sin\left(\frac{n\pi y}{\sqrt{2}}\right), \qquad \lambda_{mn} = \frac{\pi^2}{2}\left(2m^2 + n^2\right)$$

Make table for $2m^2 + n^2$,

$m \backslash n$	1	2
1	3	5
2	5	
3		

Thus 5 repeats, and both have eigenvalue $\lambda_{21} = \lambda_{12} = 5\pi^2/2$. We add

$$v = v_{21} + Av_{12}$$

We know these are both zero on the vertical and horizontal side. We find A such that they are zero on $y = \sqrt{2}x$,

$$\begin{aligned} 0 &= v\left(x, \sqrt{2}x\right) \\ &= v_{21}\left(x, \sqrt{2}x\right) + Av_{12}\left(x, \sqrt{2}x\right) \\ &= \sin(2\pi x)\sin(\pi x) + A\sin(\pi x)\sin(2\pi x) \end{aligned}$$

Thus $A = -1$.

2 Use $v(x, y) = y(x-1)\left(\sqrt{2}x - y\right)$. Use Rayleigh Quotient, will be upper bound on λ_1.