Solutions to Practice Test 1

1 Given

You may assume the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} X'' + \lambda X &= 0, \qquad 0 < x < 1 \\ X(0) &= 0 \qquad X(1) = 0 \end{aligned}$$

are $\lambda_n = n^2\pi^2$ and $X_n(x) = \sin(nx)$, for $n = 1, 2, ...$, without derivation.

You may also assume the following orthogonality conditions for m, n positive integers:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

2 Question

Consider the following heat problem in dimensionless variables

$$\begin{aligned} u_t &= u_{xx} + \frac{\pi^2}{4}u - b, \qquad 0 < x < 1, \qquad t > 0 && (1) \\ u(0,t) &= 0, \qquad u(1,t) = 0, \qquad t > 0 && (2) \\ u(x,0) &= u_0 \qquad 0 < x < 1. && (3) \end{aligned}$$

(a) [3 points] Explain in terms of a heated rod precisely what the problem models mathematically.

Solution: The problem models heat transfer in a rod of (scaled) length 1, with thermal diffusivity 1. The temperature is fixed at zero degrees at both ends and the rod is initially at a constant temperature u_0. Heat is absorbed througout the rod at a rate of b and produced/absorbed at a rate proportional to the current temperature (proportionality constant 1/4).

(b) [3 points] Derive the equilibrium solution

$$u_E(x) = \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right)$$

It is insufficient to simply verify that the solution works.

Solution: The equilibrium solution $u_E(x)$ satisfies

$$u_E''(x) + \frac{\pi^2}{4}u_E(x) = b$$

$$u_E(0) = 0 = u_E(1)$$

The ODE has solution

$$u_E(x) = A\cos\left(\frac{\pi x}{2}\right) + B\sin\left(\frac{\pi x}{2}\right) + \frac{4b}{\pi^2}$$

Imposing the BCs gives

$$\begin{aligned} u_E(0) &= A + 4b/\pi^2 = 0 \\ u_E(1) &= B + 4b/\pi^2 = 0 \end{aligned}$$

Solving for A, B gives $A = B = -4b/\pi^2$. Putting things together gives

$$u_E(x) = \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right)$$

(c) [3 points] Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$\begin{aligned} v_t &= v_{xx} + \frac{\pi^2}{4}v, \qquad 0 < x < 1, \qquad t > 0 && (4) \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 && (5) \\ v(x,0) &= f(x) \qquad 0 < x < 1. && (6) \end{aligned}$$

where $f(x)$ will be determined by the transformation.

Solution: We let

$$v(x,t) = u(x,t) - u_E(x)$$

or

$$u(x,t) = v(x,t) + u_E(x)$$

Then

$$u_t = v_t, \qquad u_{xx} = v_{xx} + u_E'' = v_{xx} + b\left(\cos\left(\frac{\pi x}{2}\right) + \sin\left(\frac{\pi x}{2}\right)\right)$$

so that the PDE (1) for $u(x,t)$ becomes

$$\begin{aligned} v_t &= v_{xx} + b\left(\cos\left(\frac{\pi x}{2}\right) + \sin\left(\frac{\pi x}{2}\right)\right) + \frac{\pi^2}{4}u_E + \frac{\pi^2}{4}v - b \\ &= v_{xx} + \frac{\pi^2}{4}v \end{aligned}$$

Thus, the PDE becomes

$$v_t = v_{xx} + \frac{\pi^2}{4}v$$

The BCs (2) become

$$\begin{aligned} v(0,t) &= u(0,t) - u_E(0) = 0 - 0 = 0 \\ v(1,t) &= u(1,t) - u_E(1) = 0 - 0 = 0 \end{aligned}$$

The IC (3) becomes

$$v(x,0) = u(x,0) - u_E(x) = u_0 - \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right)$$

We have shown that $v(x,t)$ satisfies the PDE (4), BCs (5) and the IC (6) with

$$f(x) = u_0 - \frac{4b}{\pi^2}\left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right)\right) \tag{7}$$

(d) [3 points] For an appropriate value of α show that the transformation $w(x,t) = e^{\alpha t}v(x,t)$ further simplifies the problem to

$$\begin{aligned} w_t &= w_{xx}, \qquad 0 < x < 1, \qquad t > 0 && (8) \\ w(0,t) &= 0, \qquad w(1,t) = 0, \qquad t > 0 && (9) \\ w(x,0) &= f(x) \qquad 0 < x < 1. && (10) \end{aligned}$$

Solution: Letting $w(x,t) = e^{\alpha t}v(x,t)$, the BCs (5) and IC (6) become

$$\begin{aligned} w(0,t) &= e^{\alpha t}v(0,t) = 0, \\ w(1,t) &= e^{\alpha t}v(1,t) = 0, \end{aligned}$$

$$w(x,0) = v(x,0) = f(x)$$

To transform the PDE, note that $v(x,t) = e^{-\alpha t} w(x,t)$ and hence

$$\begin{aligned} v_t &= -\alpha e^{-\alpha t} w + e^{-\alpha t} w_t \\ v_{xx} &= e^{-\alpha t} w_{xx} \end{aligned}$$

so the PDE (4) for $v(x,t)$ becomes

$$-\alpha e^{-\alpha t} w + e^{-\alpha t} w_t = e^{-\alpha t} w_{xx} + \frac{\pi^2}{4} e^{-\alpha t} w$$

Multiplying by $e^{\alpha t}$ and rearranging gives

$$w_t = w_{xx} + \left(\alpha + \frac{\pi^2}{4}\right) w$$

Choosing $\alpha = -\pi^2/4$ yields

$$w_t = w_{xx}$$

with $v(x,t) = e^{\pi^2 t/4} w(x,t)$. We have shown that $w(x,t)$ satisfies the PDE (8), BCs (9) and the IC (10) with $f(x)$ given in (7).

(e) [8 points] Derive the solution

$$w(x,t) = \sum_{n=1}^{\infty} w_n(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{2(u_0 - 4b/\pi^2)}{2n-1} + \frac{32b(2n-1)}{\pi^2 (4n-3)(4n-1)} \right) e^{-(2n-1)^2 \pi^2 t} \sin((2n-1)\pi x)$$

and hence solve for $u(x,t) = u_E(x) + \sum_{n=1}^{\infty} u_n(x,t)$ using the earlier transformations.

Solution: Note that the PDE (8), BCs (9) and the IC (10) are the basic heat problem we considered in class. We derived the solution using separation of variables,

$$w(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t} \tag{11}$$

where

$$B_n = 2\int_0^1 w(x,0) \sin(n\pi x)\, dx = 2\int_0^1 f(x) \sin(n\pi x)\, dx \tag{12}$$

and $f(x)$ is given in (7). Note that

$$\begin{aligned} \int_0^1 \sin(n\pi x)\, dx &= \frac{1}{n\pi} \left[-\cos(n\pi x)\right]_0^1 \\ &= \frac{1}{n\pi}(1 - \cos(n\pi)) = \frac{1}{n\pi}\left(1 - (-1)^n\right) \end{aligned}$$

$$
\begin{aligned}
\int_0^1 \cos\left(\frac{\pi x}{2}\right)\sin(n\pi x)\,dx &= \int_0^1 \frac{1}{2}\left(\sin\left(\frac{2n+1}{2}\pi x\right)+\sin\left(\frac{2n-1}{2}\pi x\right)\right)dx \\
&= \frac{1}{2}\left[-\frac{2\cos\left(\frac{2n+1}{2}\pi x\right)}{(2n+1)\pi}-\frac{2\cos\left(\frac{2n-1}{2}\pi x\right)}{(2n-1)\pi}\right]_0^1 \\
&= \frac{1}{(2n+1)\pi}+\frac{1}{(2n-1)\pi} \\
&= \frac{4n}{(2n+1)(2n-1)\pi}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 \sin\left(\frac{\pi x}{2}\right)\sin(n\pi x)\,dx &= \int_0^1 \frac{1}{2}\left(-\cos\left(\frac{2n+1}{2}\pi x\right)+\cos\left(\frac{2n-1}{2}\pi x\right)\right)dx \\
&= \frac{1}{2}\left[-\frac{2\sin\left(\frac{2n+1}{2}\pi x\right)}{(2n+1)\pi}+\frac{2\sin\left(\frac{2n-1}{2}\pi x\right)}{(2n-1)\pi}\right]_0^1 \\
&= -\frac{\sin\left(\frac{2n+1}{2}\pi\right)}{(2n+1)\pi}+\frac{\sin\left(\frac{2n-1}{2}\pi\right)}{(2n-1)\pi} \\
&= -\frac{(-1)^n}{(2n+1)\pi}+\frac{(-1)^{n+1}}{(2n-1)\pi} \\
&= -\frac{4n(-1)^n}{(2n+1)(2n-1)\pi}
\end{aligned}
$$

Thus (12) becomes

$$
\begin{aligned}
B_n &= 2\int_0^1 f(x)\sin(n\pi x)\,dx \\
&= 2\int_0^1 \left(u_0-\frac{4b}{\pi^2}\left(1-\cos\left(\frac{\pi x}{2}\right)-\sin\left(\frac{\pi x}{2}\right)\right)\right)\sin(n\pi x)\,dx \\
&= 2\left(u_0-\frac{4b}{\pi^2}\right)\int_0^1 \sin(n\pi x)\,dx \\
&\quad +\frac{8b}{\pi^2}\int_0^1 \left(\cos\left(\frac{\pi x}{2}\right)+\sin\left(\frac{\pi x}{2}\right)\right)\sin(n\pi x)\,dx \\
&= \frac{2}{n\pi}\left(u_0-\frac{4b}{\pi^2}\right)(1-(-1)^n)+\frac{16bn(1-(-1)^n)}{\pi^3(2n+1)(2n-1)} \\
&= \begin{cases} \frac{4\left(u_0-4b/\pi^2\right)}{(2m-1)\pi}+\frac{32b(2m-1)}{(4m-1)(4m-3)\pi^2}, & n=2m-1 \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}
$$

Substituting B_n into (11) gives

$$
w(x,t)=\sum_{m=1}^{\infty}\frac{2}{\pi}\left(\frac{2\left(u_0-4b/\pi^2\right)}{2m-1}+\frac{32b(2m-1)}{\pi^2(4m-1)(4m-3)}\right)\sin((2m-1)\pi x)\,e^{-(2m-1)^2\pi^2 t}
$$

as required. The solution $u(x,t)$ is given by reversing our transformations,

$$
\begin{aligned}
u(x,t) &= e^{\pi^2 t/4} w(x,t) + u_E(x) \\
&= e^{\pi^2 t/4} \sum_{m=1}^{\infty} \frac{2}{\pi} \left(\frac{2\left(u_0 - 4b/\pi^2\right)}{2m-1} + \frac{32b\,(2m-1)}{\pi^2\,(4m-1)\,(4m-3)} \right) \sin\left((2m-1)\,\pi x\right) e^{-(2m-1)^2 \pi^2 t} \\
&\quad + \frac{4b}{\pi^2} \left(1 - \cos\left(\frac{\pi x}{2}\right) - \sin\left(\frac{\pi x}{2}\right) \right)
\end{aligned}
$$

Aside (optional): a quick check of the above formula for $w(x,t)$:

1. $w(0,t) = 0 = w(1,t)$
2. $w(x,0) =$ fourier series of $f(x)$
3. $w_t = w_{xx}$ since $\sin\left((2m-1)\,\pi x\right) e^{-(2m-1)^2 \pi^2 t}$ satisfies the PDE for all m.