1 Given

You may assume the eigenvalues of the Sturm-Liouville problem

\[ X'' + \lambda X = 0, \quad 0 < x < 1 \]
\[ X(0) = 0 \quad X(1) = 0 \]

are \( \lambda_n = n^2 \pi^2 \) and \( X_n(x) = \sin(nx) \), for \( n = 1, 2, \ldots \), without derivation.

You may also assume the following orthogonality conditions for \( m, n \) positive integers:

\[
\int_0^1 \sin(m \pi x) \sin(n \pi x) \, dx = \begin{cases} 
1/2, & m = n \neq 0, \\
0, & m \neq n.
\end{cases}
\]

\[
\int_0^1 \cos(m \pi x) \cos(n \pi x) \, dx = \begin{cases} 
1/2, & m = n \neq 0, \\
0, & m \neq n.
\end{cases}
\]

2 Question

Consider the following heat problem in dimensionless variables

\[ u_t = u_{xx} + \frac{\pi^2}{4} u - b, \quad 0 < x < 1, \quad t > 0 \quad (1) \]
\[ u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0 \quad (2) \]
\[ u(x,0) = u_0 \quad 0 < x < 1. \quad (3) \]

(a) [3 points] Explain in terms of a heated rod precisely what the problem models mathematically.
Solution: The problem models heat transfer in a rod of (scaled) length 1, with thermal diffusivity 1. The temperature is fixed at zero degrees at both ends and the rod is initially at a constant temperature $u_0$. Heat is absorbed throughout the rod at a rate of $b$ and produced/absorbed at a rate proportional to the current temperature (proportionality constant $1/4$).

(b) [3 points] Derive the equilibrium solution

$$u_E(x) = \frac{4b}{\pi^2} \left(1 - \cos \left(\frac{\pi x}{2}\right) - \sin \left(\frac{\pi x}{2}\right)\right)$$

It is insufficient to simply verify that the solution works.

Solution: The equilibrium solution $u_E(x)$ satisfies

$$u''_E(x) + \frac{\pi^2}{4} u_E(x) = b$$

$$u_E(0) = 0 = u_E(1)$$

The ODE has solution

$$u_E(x) = A \cos \left(\frac{\pi x}{2}\right) + B \sin \left(\frac{\pi x}{2}\right) + \frac{4b}{\pi^2}$$

Imposing the BCs gives

$$u_E(0) = A + 4b/\pi^2 = 0$$

$$u_E(1) = B + 4b/\pi^2 = 0$$

Solving for $A, B$ gives $A = B = -4b/\pi^2$. Putting things together gives

$$u_E(x) = \frac{4b}{\pi^2} \left(1 - \cos \left(\frac{\pi x}{2}\right) - \sin \left(\frac{\pi x}{2}\right)\right)$$

(c) [3 points] Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$v_t = v_{xx} + \frac{\pi^2}{4} v, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = f(x) \quad 0 < x < 1.$$  

where $f(x)$ will be determined by the transformation.

Solution: We let

$$v(x,t) = u(x,t) - u_E(x)$$

2
\[ u(x, t) = v(x, t) + u_E(x) \]

Then
\[ u_t = v_t, \quad u_{xx} = v_{xx} + u''_E = v_{xx} + b \left( \cos \left( \frac{\pi x}{2} \right) + \sin \left( \frac{\pi x}{2} \right) \right) \]

so that the PDE (1) for \( u(x, t) \) becomes
\[
\begin{align*}
v_t &= v_{xx} + b \left( \cos \left( \frac{\pi x}{2} \right) + \sin \left( \frac{\pi x}{2} \right) \right) + \frac{\pi^2}{4} u_E + \frac{\pi^2}{4} v - b \\
&= v_{xx} + \frac{\pi^2}{4} v
\end{align*}
\]

Thus, the PDE becomes
\[ u_t = v_{xx} + \frac{\pi^2}{4} v \]

The BCs (2) become
\[
\begin{align*}
v(0, t) &= u(0, t) - u_E(0) = 0 - 0 = 0 \\
v(1, t) &= u(1, t) - u_E(1) = 0 - 0 = 0
\end{align*}
\]

The IC (3) becomes
\[
v(x, 0) = u(x, 0) - u_E(x) = u_0 - \frac{4b}{\pi^2} \left( 1 - \cos \left( \frac{\pi x}{2} \right) - \sin \left( \frac{\pi x}{2} \right) \right)
\]

We have shown that \( v(x, t) \) satisfies the PDE (4), BCs (5) and the IC (6) with
\[
f(x) = u_0 - \frac{4b}{\pi^2} \left( 1 - \cos \left( \frac{\pi x}{2} \right) - \sin \left( \frac{\pi x}{2} \right) \right) \tag{7}
\]

(d) [3 points] For an appropriate value of \( \alpha \) show that the transformation \( w(x, t) = e^{\alpha t} v(x, t) \) further simplifies the problem to
\[
\begin{align*}
w_t &= w_{xx}, \quad 0 < x < 1, \quad t > 0 \\
w(0, t) &= 0, \quad w(1, t) = 0, \quad t > 0 \\
w(x, 0) &= f(x) \quad 0 < x < 1.
\end{align*}
\]

**Solution:** Letting \( w(x, t) = e^{\alpha t} v(x, t) \), the BCs (5) and IC (6) become
\[
\begin{align*}
w(0, t) &= e^{\alpha t} v(0, t) = 0, \\
w(1, t) &= e^{\alpha t} v(1, t) = 0, \\
w(x, 0) &= v(x, 0) = f(x)
\end{align*}
\]
To transform the PDE, note that $v(x, t) = e^{-\alpha t} w(x, t)$ and hence

$$
v_t = -\alpha e^{-\alpha t} w + e^{-\alpha t} w_t
$$

$$
v_{xx} = e^{-\alpha t} w_{xx}
$$

so the PDE (4) for $v(x, t)$ becomes

$$
-\alpha e^{-\alpha t} w + e^{-\alpha t} w_t = e^{-\alpha t} w_{xx} + \frac{\pi^2}{4} e^{-\alpha t} w
$$

Multiplying by $e^{\alpha t}$ and rearranging gives

$$
w_t = w_{xx} + \left( \alpha + \frac{\pi^2}{4} \right) w
$$

Choosing $\alpha = -\pi^2/4$ yields

$$
w_t = w_{xx}
$$

with $v(x, t) = e^{\pi^2 t/4} w(x, t)$. We have shown that $w(x, t)$ satisfies the PDE (8), BCs (9) and the IC (10) with $f(x)$ given in (7).

(e) [8 points] Derive the solution

$$
w(x, t) = \sum_{n=1}^{\infty} w_n(x, t) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left( \frac{2(u_0 - 4b/\pi^2)}{2n - 1} + \frac{32b(2n - 1)}{\pi^2(4n - 3)(4n - 1)} \right) e^{-(2n-1)^2\pi^2 t} \sin((2n - 1)\pi x)
$$

and hence solve for $u(x, t) = u_E(x) + \sum_{n=1}^{\infty} u_n(x, t)$ using the earlier transformations.

**Solution:** Note that the PDE (8), BCs (9) and the IC (10) are the basic heat problem we considered in class. We derived the solution using separation of variables,

$$
w(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t}
$$

(11)

where

$$
B_n = 2 \int_0^1 w(x, 0) \sin(n\pi x) \, dx = 2 \int_0^1 f(x) \sin(n\pi x) \, dx
$$

(12)

and $f(x)$ is given in (7). Note that

$$
\int_0^1 \sin(n\pi x) \, dx = \frac{1}{n\pi} [\cos(n\pi)]_0^1
$$

$$
= \frac{1}{n\pi} (1 - \cos(n\pi)) = \frac{1}{n\pi} (1 - (-1)^n)
$$
\[
\int_0^1 \cos \left( \frac{\pi x}{2} \right) \sin (n \pi x) \, dx = \int_0^1 \frac{1}{2} \left( \sin \left( \frac{2n+1}{2} \pi x \right) + \sin \left( \frac{2n-1}{2} \pi x \right) \right) \, dx \\
= \frac{1}{2} \left[ \frac{2 \cos \left( \frac{2n+1}{2} \pi x \right)}{(2n+1) \pi} - \frac{2 \cos \left( \frac{2n-1}{2} \pi x \right)}{(2n-1) \pi} \right]_0^1 \\
= \frac{1}{(2n+1) \pi} + \frac{1}{(2n-1) \pi} \\
= \frac{4n}{(2n+1) (2n-1) \pi}
\]

\[
\int_0^1 \sin \left( \frac{\pi x}{2} \right) \sin (n \pi x) \, dx = \int_0^1 \frac{1}{2} \left( -\cos \left( \frac{2n+1}{2} \pi x \right) + \cos \left( \frac{2n-1}{2} \pi x \right) \right) \, dx \\
= \frac{1}{2} \left[ \frac{2 \sin \left( \frac{2n+1}{2} \pi x \right)}{(2n+1) \pi} + \frac{2 \sin \left( \frac{2n-1}{2} \pi x \right)}{(2n-1) \pi} \right]_0^1 \\
= -\frac{\sin \left( \frac{2n+1}{2} \pi \right)}{(2n+1) \pi} + \frac{\sin \left( \frac{2n-1}{2} \pi \right)}{(2n-1) \pi} \\
= -\frac{(-1)^n}{(2n+1) \pi} + \frac{(-1)^{n+1}}{(2n-1) \pi} \\
= -\frac{4n (-1)^n}{(2n+1) (2n-1) \pi}
\]

Thus (12) becomes

\[
B_n = 2 \int_0^1 f(x) \sin (n \pi x) \, dx \\
= 2 \int_0^1 \left( u_0 - \frac{4b}{\pi} \left( 1 - \cos \left( \frac{\pi x}{2} \right) - \sin \left( \frac{\pi x}{2} \right) \right) \right) \sin (n \pi x) \, dx \\
= 2 \left( u_0 - \frac{4b}{\pi^2} \right) \int_0^1 \sin (n \pi x) \, dx \\
+ \frac{8b}{\pi^2} \int_0^1 \left( \cos \left( \frac{\pi x}{2} \right) + \sin \left( \frac{\pi x}{2} \right) \right) \sin (n \pi x) \, dx \\
= \frac{2}{n \pi} \left( u_0 - \frac{4b}{\pi^2} \right) (1 - (-1)^n) + \frac{16bn (1 - (-1)^n)}{\pi^3 (2n+1) (2n-1)} \\
= \begin{cases} \\
\frac{4(u_0-4b/\pi^2)}{(2m-1)\pi} + \frac{32b(2m-1)}{(4m-1)(4m-3)\pi^2}, & n = 2m - 1 \text{ odd} \\
0, & n = 2m \text{ even}
\end{cases}
\]

Substituting $B_n$ into (11) gives

\[
w(x, t) = \sum_{m=1}^{\infty} \frac{2}{\pi} \left( \frac{2(u_0 - 4b/\pi^2)}{2m - 1} + \frac{32b(2m - 1)}{\pi^2 (4m - 1)(4m - 3)} \right) \sin ((2m - 1) \pi x) e^{-(2m-1)^2 \pi^2 t}
\]
as required. The solution \( u(x, t) \) is given by reversing our transformations,

\[
u(x, t) = e^{\pi^2 t/4} w(x, t) + u_E(x)
\]

\[
e^{\pi^2 t/4} \sum_{m=1}^{\infty} \frac{2}{\pi} \left( \frac{2 (u_0 - 4b/\pi^2)}{2m - 1} + \frac{32b (2m - 1)}{\pi^2 (4m - 1) (4m - 3)} \right) \sin ((2m - 1) \pi x) e^{-(2m-1)^2 \pi^2 t}
\]

\[
+ \frac{4b}{\pi^2} \left( 1 - \cos \left( \frac{\pi x}{2} \right) - \sin \left( \frac{\pi x}{2} \right) \right)
\]

Aside (optional): a quick check of the above formula for \( w(x, t) \):

1. \( w(0, t) = 0 = w(1, t) \)
2. \( w(x, 0) = \text{fourier series of } f(x) \)
3. \( w_t = w_{xx} \) since \( \sin ((2m - 1) \pi x) e^{-(2m-1)^2 \pi^2 t} \) satisfies the PDE for all \( m \).