Test 1
Introduction to PDE
MATH 3363-25820 (Fall 2009)

Solutions to Test 1

This exam has 4 questions, for a total of 20 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Upon finishing PLEASE write and sign your pledge below:
On my honor I have neither given nor received any aid on this exam.

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1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided. Be sure to show a few key intermediate steps when deriving results - answers only will not get full marks.

2 Given

You may assume the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} X'' + \lambda X = 0, &\quad 0 < x < 1 \\ X'(0) = 0, &\quad X(1) = 0. \end{aligned}$$

are $\lambda_n = (n - \frac{1}{2})^2\pi^2$ and $X_n(x) = \cos\big((n - \frac{1}{2})\pi x\big)$, for $n = 1, 2, \ldots$, without derivation.

You may also assume the following orthogonality conditions for m, n positive integers:

$$\int_0^1 \cos\big((m - \frac{1}{2})\pi x\big)\cos\big((n - \frac{1}{2})\pi x\big)dx = \begin{cases} 1/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

3 Questions

Consider the following heat problem in dimensionless variables

$$u_t = u_{xx} + bx^2, \quad 0 < x < 1, \quad t > 0$$
$$u_x(0,t) = 0, \quad u(1,t) = 1, \quad t > 0$$
$$u(x,0) = u_0, \quad 0 < x < 1,$$

where $b > 0$ and $u_0 > 0$ are constants. This is the heat equation with a source, where the rod is insulated at $x = 0$ and kept at 1 degree at $x = 1$.

4 points 1. Derive the steady-state (equilibrium) solution

$$u_E(x) = \frac{b}{12}(1 - x^4) + 1$$

It is insufficient to simply verify that the solution works.

Solutoin: The steady-state solution $u_E(x)$ satisfies the PDE and BCs, yielding

[1 point] $$0 = u_E'' + bx^2, \quad 0 < x < 1$$
[1 point] $$u_E'(0) = 0, \quad u_E(1) = 1$$

[1 point] Integrating (twice) the ODE, i.e., $u_E'' = -bx^2$, for u_E gives

$$u_E(x) = -\frac{b}{12}x^4 + C_1 x + C_2$$

[1 point] Imposing the BCs: $u_E'(0) = 0 \Rightarrow C_1 = 0$ and $u_E(1) = 1 \Rightarrow C_2 = \frac{b}{12} + 1$, yielding

$$u_E(x) = \frac{b}{12}(1 - x^4) + 1$$

3 points 2. (a) Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0$$
$$v_x(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$
$$v(x,0) = f(x), \quad 0 < x < 1,$$

where $f(x)$ will be determined by the transformation.
Show your work, which involves writing $v = u - u_E$ and using the information from u

and u_E to derive the problem for v.

(b) State $f(x)$ in terms of u_0, b and x.

Solutoin: Writing

[0.5 points]
$$v(x,t) = u(x,t) - u_E(x)$$

we have

[0.5 points]
$$v_t = u_t$$

[0.5 points]
$$v_{xx} = u_{xx} - u_E'' = u_{xx} + bx^2$$

[0.5 points] and hence the PDE becomes

$$v_t = v_{xx}$$

[0.5 points] The BCs for v are

$$v_x(0,t) = u_x(0,t) - u_E'(0) = 0 - 0 = 0$$
$$v(1,t) = u(1,t) - u_E(1) = 1 - 1 = 0$$

[0.5 points] The IC is

$$v(x,0) = u(x,0) - u_E(x) = u_0 - \frac{b}{12}(1 - x^4) - 1$$

thus

$$f(x) = u_0 - 1 - \frac{b}{12}(1 - x^4).$$

$\boxed{\text{10 points}}$ 3. Derive the solution

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} A_n \cos\big((n - \frac{1}{2})\pi x\big)\, e^{-(n-\frac{1}{2})^2\pi^2 t}$$

and derive equations for A_n in terms of $f(x)$. Be sure to give the intermediate steps: (a) separate variables, (b) write down problems and solve for $X(x)$ (using information from the Given section), (c) solve for $T_n(t)$, (d) put things together, impose the IC, (e) use orthogonality of $\cos\big((n - \frac{1}{2})\pi x\big)$ (see Given section) to find A_n in terms of $f(x)$. Substitute for $f(x)$ from Part 2. You may use (without proof) the following integrals, for any integer n,

$$\int_0^1 (1 - x^4)\cos\big((n - \frac{1}{2})\pi x\big)dx = \frac{(-1)^{n+1}96(\pi^2(2n-1)^2 - 8)}{(2n-1)^5\pi^5},$$

$$\int_0^1 \cos\big((n - \frac{1}{2})\pi x\big)dx = \frac{2(-1)^{n+1}}{(2n-1)\pi}$$

Solutoin: Using separation of variables, we let

[0.5 points] $$v(x,t) = X(x)T(t)$$

and substitute this into the PDE to obtain

[1 point] $$\frac{X''}{X} = \frac{T''}{T} = -\lambda$$

where λ is a constant because the left hand side depends only on x and the middle only depends on t.
The Sturm-Liouville problem for $X(x)$ is

[1 point] $$X'' + \lambda X = 0; \quad X'(0) = 0 = X(1)$$

whose solution is (given),

[0.5 points] $$X_n(x) = \cos\big((n - \frac{1}{2})\pi x\big), \quad \lambda_n = (n - \frac{1}{2})^2\pi^2, \quad n = 1, 2, \ldots$$

The equations for $T(t)$ are

[1 point] $$T_n(t) = A_n e^{-(n-\frac{1}{2})^2\pi^2 t}$$

and this gives the solution $v_n(x,t)$ to the PDE

[0.5 points] $$v_n(x,t) = X_n(x)T_n(t) = A_n \cos\big((n - \frac{1}{2})\pi x\big)\, e^{-(n-\frac{1}{2})^2\pi^2 t}$$

for constants A_n. Summing all $v_n(x,t)$ together gives

[1 point] $$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} A_n \cos\big((n - \frac{1}{2})\pi x\big) e^{-(n-\frac{1}{2})^2\pi^2 t}$$

Imposing the IC gives

[1 point] $$v(x,0) = \sum_{n=1}^{\infty} A_n \cos\big((n - \frac{1}{2})\pi x\big)$$

Multiplying by $\cos\big((m - \frac{1}{2})\pi x\big)$ and integrating from $x = 0$ to $x = 1$ gives

[1 point] $$\int_0^1 f(x)\cos\big((m - \frac{1}{2})\pi x\big)dx = \sum_{n=1}^{\infty} A_n \int_0^1 \cos\big((n - \frac{1}{2})\pi x\big)\cos\big((m - \frac{1}{2})\pi x\big)dx$$

Using the given orthogonality condition gives

[1 point] $$A_m = 2\int_0^1 f(x)\cos\big((m - \frac{1}{2})\pi x\big)dx$$

Substituting for $f(x)$ gives

[0.5 points] $$A_m = 2\int_0^1 \Big(u_0 - 1 - \frac{b}{12}(1-x^4)\Big)\cos\big((m-\frac{1}{2})\pi x\big)\,dx$$

[0.5 points] $$= 2(u_0-1)\int_0^1 \cos\big((m-\frac{1}{2})\pi x\big)\,dx - \frac{b}{6}\int_0^1 (1-x^4)\cos\big((m-\frac{1}{2})\pi x\big)\,dx$$

[0.5 points] $$= \frac{(-1)^{n+1}4(u_0-1)}{(2n-1)\pi} - \frac{(-1)^{n+1}16b(\pi^2(2n-1)^2-8)}{(2n-1)^5\pi^5}$$

$\boxed{\text{3 points}}$ 4. (a) Solve for

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} u_n(x,t)$$

using the earlier transformations. Write down precisely the functions $u_n(x,t)$.

(b) Find an approximate solution good for large time and state the limit as $t \to \infty$.

Solutoin: Reversing the earlier transformations, we have

[0.5 points] $$u(x,t) = u_E(x) + v(x,t)$$
$$= \frac{b}{12}(1-x^4) + 1 + \sum_{n=1}^{\infty} A_n \cos\big((n-\frac{1}{2})\pi x\big)e^{-(n-\frac{1}{2})^2\pi^2 t}$$

thus

[0.5 points] $$u_n(x,t) = A_n\cos\big((n-\frac{1}{2})\pi x\big)e^{-(n-\frac{1}{2})^2\pi^2 t}$$

where A_n is given in Part 3. For $t > 1/\pi^2$, the first term in the series gives a good approximation to $v(x,t)$, thus

[1 point] $$u(x,t) \approx u_E(x) + u_1(x,t)$$
$$= \frac{b}{12}(1-x^4) + 1 + \left(\frac{4(u_0-1)}{\pi} - \frac{16b(\pi^2-8)}{\pi^5}\right)\cos\big(\frac{1}{2}\pi x\big)e^{-\frac{1}{4}\pi^2 t}$$

We have

[1 point] $$u(x,t) \to u_E(x) \quad \text{as } t\to\infty$$