Test 2
Introduction to PDE
MATH 3363-25820 (Fall 2009)

Name and UH-ID:

This exam has 2 questions, for a total of 20 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Upon finishing PLEASE write and sign your pledge below:
On my honor I have neither given nor received any aid on this exam.

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1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided. Be sure to show a few key intermediate steps when deriving results - answers only will not get full marks.

2 Given

The solution to the 1d wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

subject to the ICs

$$u(x,0) = f(x), \quad -\infty < x < \infty,$$
$$u_t(x,0) = g(x), \quad -\infty < x < \infty,$$

is

$$u(x,t) = F(x-ct) + G(x+ct)$$

where

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds$$
$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds$$

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3 Questions

8 points

1. Suppose that an "infinite string" is initially streched into the shape of a single rectangular pulse and is let go from the rest. We model the problem using the 1D wave equation

$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

subject to the initial conditions

$$u(x,0) = f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} = \begin{cases} 0, & x < -1 \\ 1, & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

$$u_t(x,0) = 0.$$

(a) Plot the four characteristics: $x - t = -1$, $x + t = -1$, $x - t = 1$ and $x + t = 1$ in the space-time plane (xt). Show that these four characteristics divide the space-time plane (xt) into six distinct regions. In each region, show that the solution u is constant, and give its value.

(b) Determine analytic formulas for $u(x,t)$ for $t < 1$ and $t > 1$.

(c) Illustrate the time evolution of the string by sketching the ux-profiles $u(x,t)$ of the string displacement for $t = 0$, $1/2$, 1, $3/2$.

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12 points 2. Consider the quasi-linear PDE

$$\frac{\partial u}{\partial t} + (1+u)\frac{\partial u}{\partial x} = 0, \quad u(x,0) = f(x)$$

where the initial condition $f(x)$ is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \le x \le 0 \\ 2 - x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(a) By writing the PDE in the form $(A, B, C) \cdot (\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1) = 0$, find the parametric solution using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x). First write down the relevant ODEs for $\frac{dt}{dr}$, $\frac{dx}{dr}$, $\frac{du}{dr}$. Take the initial conditions $t = 0$ and $x = s$ at $r = 0$. Using the initial condition, write down the IC for u at $r = 0$, in terms of s. Solve for t, u and x as functions of r, s.

(b) At what time t_{sh} and location x_{sh} does your parametric solution break down?

(c) For each of $s = -1, 0, 1$, write down the characteristic x as a functions of t and plot the characteristic in the space-time plane (xt).

(d) The tables below are useful as a plotting aid for (e). Fill in the tables using your result from (a) or (c) to obtain u and x at the s-values listed at time $t = 1/2, 1$.

$t = \frac{1}{2}$	$s =$	-1	0	1
	$u =$			
	$x =$			

$t = 1$	$s =$	-1	0	1
	$u =$			
	$x =$			

(e) Illustrate the time evolution of the solution by sketching the ux-profiles $u(x, t)$ for $t = 0,\ 1/2,\ 1$.

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1.

1 cont.

2.

2 cont.

2 cont.