Problems for the 1-D Wave Equation

1 Problem 1

(i) Suppose that an "infinite string" has an initial displacement

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-2x, & 0 \le x \le 1/2 \\ 0, & x < -1 \text{ and } x > 1/2 \end{cases}$$

and zero initial velocity $u_t(x,0) = 0$. Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs $u(x,0) = f(x)$ and $u_t(x,0) = 0$ using D'Alembert's formula. Illustrate the nature of the solution by sketching the ux-profiles $y = u(x,t)$ of the string displacement for $t = 0, 1/2, 1, 3/2$.

(ii) Repeat the procedure in (i) for a string that has zero initial displacement but is given an initial velocity

$$u_t(x,0) = g(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \\ 0, & x < -1 \text{ and } x > 1 \end{cases}$$

3 Question 1

[20 points total]

Suppose you shake the end of a rope of dimensionless length 1 at a certain frequency ω. We neglect gravity and friction and model the waves on the rope using the 1D wave equation:

$$u_{tt} = u_{xx}, \qquad 0 < x < 1, \qquad t > 0 \tag{1}$$

where u is the displacement of the rope away from its rest state. The rope is attached and held fixed at a wall at $x = 0$,

$$u(0,t) = 0, \qquad t > 0. \tag{2}$$

You shake the other end ($x = 1$) sinusoidally, with frequency ω, and give it the displacement

$$u(1,t) = \sin \omega t, \qquad t > 0. \tag{3}$$

We assume the rope has zero initial position and velocity

$$u(x,0) = 0, \qquad 0 < x < 1, \tag{4}$$

$$u_t(x,0) = 0, \qquad 0 < x < 1. \tag{5}$$

Note: D'Alembert's solution does not hold for u, because of the BC at $x = 1$.

(a) [10 points] Find a solution of the form

$$U(x,t) = X(x) \sin \omega t \tag{6}$$

that satisfies the PDE (1) and the BCs (2) and (3). (Don't worry about the ICs yet.) Where is the rope stationary (i.e. $U(x,t) = 0$)? For what values of ω is your solution invalid?

(b) [10 points] Use $U(x,t)$ from part (a) to find the full solution $u(x,t)$ to the PDE (1), the BCs (2) and (3), and the ICs (4) and (5). Hint: define a function $v(x,t)$ in terms of $U(x,t)$ and $u(x,t)$ so that $v(x,t)$ satisfies the PDE (1) and has $v(0,t) = 0 = v(1,t)$ (type I zero BCs). Then write down D'Alembert's solution (without derivation) to satisfy the PDE and initial conditions on $v(x,t)$ (don't evaluate the integral in D'Alembert's solution). Then adjust D'Alembert's solution to handle the zero BCs on $v(x,t)$ at $x = 0, 1$. You don't need to evaluate the integral.

4 Question 2

[30 points total]

Consider the following quasi-linear PDE,

$$\frac{\partial u}{\partial t} + (1+2u)\frac{\partial u}{\partial x} = -u; \qquad u(x,0) = f(x) \tag{7}$$

where the initial condition is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \le x \le 0 \\ 2 - x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(a) [8 points] By writing the PDE in the form $(A, B, C) \cdot (u_t, u_x, -1) = 0$, where $C = -u$, find the parametric solution using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x). First write down the relevant ODEs for $\partial t/\partial r$, $\partial x/\partial r$, $\partial u/\partial r$. Take the initial conditions $t = 0$ and $x = s$ at $r = 0$. Using the initial condition in (7), write down the IC for u at $r = 0$, in terms of s. Solve for t, u and x (in that order!) as functions of r, s. When integrating for x, be careful: u depends on r!

(b) [8 points] At what time t_{sh} and position x_{sh} does your parametric solution break down? Hint: you might need to consider negative values of $f'(s)$.

(c) [4 points] Write down x in terms of t, s and $f(s)$. For each of $s = -1$, 0, 1, write down x as a function of t.

(d) [4 points] Fill in the table below, using your result from either (a) or (c) to obtain u and x at the $s-$values listed at time $t = \ln 2$ (note that $e^{-\ln 2} = \frac{1}{2}$, and you may use $\ln 2 = 0.7$):

$t = \ln 2$

$s =$	-1	0	1
$u =$			
$x =$			

(e) [6 points] Plot $f(x)$ vs. x. Then, on the SAME plot, plot $u(x,t)$ at $t = \ln 2$ by plotting the three points (x, u) from the table in part (d) and connecting the dots.