# Solution to Problems for the 1-D Wave Equation

## 1 Problem 1

(i) Suppose that an "infinite string" has an initial displacement

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-2x, & 0 \le x \le 1/2 \\ 0, & x < -1 \text{ and } x > 1/2 \end{cases}$$

and zero initial velocity $u_t(x,0) = 0$. Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs $u(x,0) = f(x)$ and $u_t(x,0) = 0$ using D'Alembert's formula. Illustrate the nature of the solution by sketching the $ux$-profiles $y = u(x,t)$ of the string displacement for $t = 0, 1/2, 1, 3/2$.

**Solution:** D'Alembert's formula is

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(s)\,ds\right)$$

In this case $g(s) = 0$ so that

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) \tag{1}$$

The problem reduces to adding shifted copies of $f(x)$ and then plotting the associated $u(x,t)$. To determine where the functions overlap or where $u(x,t)$ is zero, we plot the characteristics $x \pm t = -1$ and $x \pm t = 1/2$ in the space time plane $(xt)$ in Figure 1.

For $t = 0$, (1) becomes

$$u(x,0) = \frac{1}{2}\left(f(x) + f(x)\right) = f(x)$$

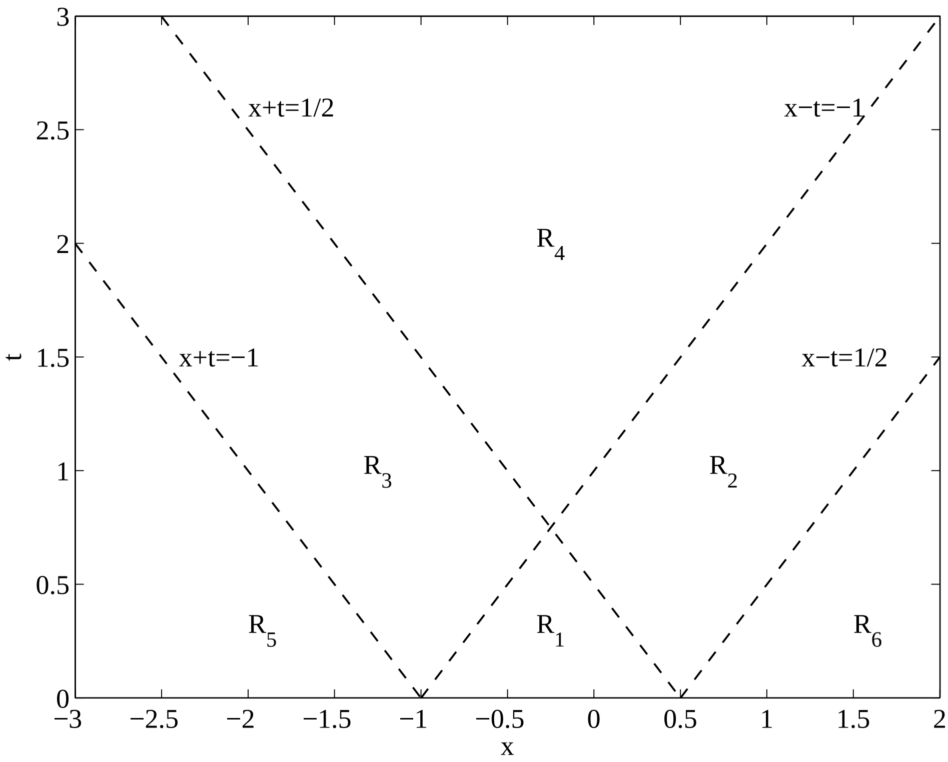


Figure 1: Sketch of characteristics for 1(a).

For $t = 1/2$, (1) becomes

$$u(x,t) = \frac{1}{2}\left(f\left(x-\frac{1}{2}\right) + f\left(x+\frac{1}{2}\right)\right)$$

Note that

$$\begin{aligned} f\left(x-\frac{1}{2}\right) &= \begin{cases} \left(x-\frac{1}{2}\right)+1, & -1 \le \left(x-\frac{1}{2}\right) \le 0 \\ 1-2\left(x-\frac{1}{2}\right), & 0 \le \left(x-\frac{1}{2}\right) \le 1/2 \\ 0, & \left(x-\frac{1}{2}\right) < -1 \text{ and } \left(x-\frac{1}{2}\right) > 1/2 \end{cases} \\ &= \begin{cases} x+\frac{1}{2}, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 2-2x, & \frac{1}{2} \le x \le 1 \\ 0, & x < -\frac{1}{2} \text{ and } x > 1 \end{cases} \end{aligned}$$

and similarly,

$$f\left(x+\frac{1}{2}\right) = \begin{cases} x+\frac{3}{2}, & -\frac{3}{2} \le x \le -\frac{1}{2} \\ -2x, & -\frac{1}{2} \le x \le 0 \\ 0, & x < -\frac{3}{2} \text{ and } x > 0 \end{cases}$$

Thus, over the region $-\frac{1}{2} \le x \le 0$ we have to be careful about adding the two functions; in the other regions either one or both functions are zero. We have

$$\begin{aligned} u\left(x,\frac{1}{2}\right) &= \frac{1}{2}\left(f\left(x-\frac{1}{2}\right)+f\left(x+\frac{1}{2}\right)\right) \\ &= \begin{cases} \frac{x}{2}+\frac{3}{4}, & -\frac{3}{2}\le x\le -\frac{1}{2} \\ -\frac{x}{2}+\frac{1}{4}, & -\frac{1}{2}\le x\le 0 \\ \frac{x}{2}+\frac{1}{4}, & 0\le x\le \frac{1}{2} \\ 1-x, & \frac{1}{2}\le x\le 1 \\ 0, & x<-\frac{3}{2} \text{ and } x>1 \end{cases} \end{aligned}$$

For $t=1$, your plot of the characteristics shows that $f(x-1)$ and $f(x+1)$ do not overlap, so you just have to worry about the different regions. Note that

$$\begin{aligned} f(x+1) &= \begin{cases} (x+1)+1, & -1\le x+1\le 0 \\ 1-2(x+1), & 0\le x+1\le 1/2 \\ 0, & x+1<-1 \text{ and } x+1>1/2 \end{cases} \\ &= \begin{cases} x+2, & -2\le x\le -1 \\ -1-2x, & -1\le x\le -1/2 \\ 0, & x<-2 \text{ and } x>-1/2 \end{cases} \end{aligned}$$

$$f(x-1)=\begin{cases} x, & 0\le x\le 1 \\ 3-2x, & 1\le x\le 3/2 \\ 0, & x<0 \text{ and } x>3/2 \end{cases}$$

so that

$$\begin{aligned} u(x,1) &= \frac{1}{2}\left(f(x-1)+f(x+1)\right) \\ &= \begin{cases} \frac{x}{2}+1, & -2\le x\le -1 \\ -\frac{1}{2}-x, & -1\le x\le -1/2 \\ \frac{x}{2}, & 0\le x\le 1 \\ \frac{3}{2}-x, & 1\le x\le 3/2 \\ 0, & x<-2,\quad -1/2<x<0, \text{ and } x>3/2 \end{cases} \end{aligned}$$

For $t=3/2$, the forward and backward waves are even further apart, and

$$f\left(x-\frac{3}{2}\right)=\begin{cases} x-\frac{1}{2}, & \frac{1}{2}\le x\le \frac{3}{2} \\ 4-2x, & \frac{3}{2}\le x\le 2 \\ 0, & x<\frac{1}{2} \text{ and } x>2 \end{cases}$$

$$f\left(x+\frac{3}{2}\right)=\begin{cases} x+\frac{5}{2}, & -\frac{5}{2}\le x\le -\frac{3}{2} \\ -2-2x, & -\frac{3}{2}\le x\le -1 \\ 0, & x<-\frac{5}{2} \text{ and } x>-1 \end{cases}$$

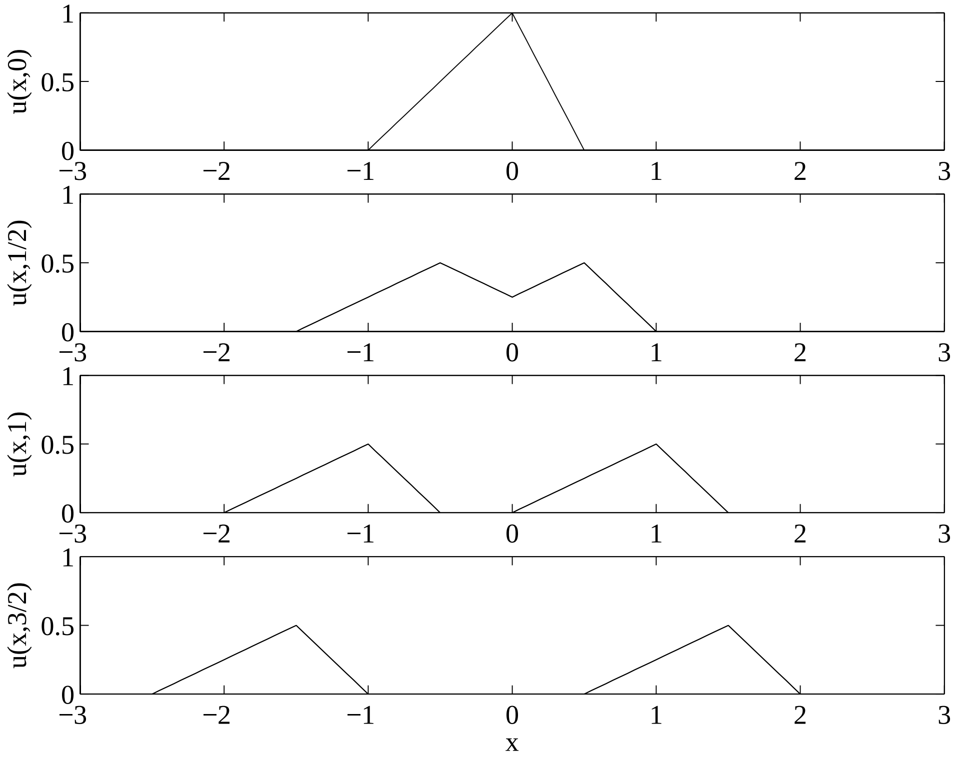


Figure 2: Plot of $u(x,t_0)$ for $t_0 = 0, 1/2, 1, 3/2$ for 1(a).

and hence

$$\begin{aligned} u\left(x,\frac{3}{2}\right) &= \frac{1}{2}\left(f\left(x-\frac{3}{2}\right)+f\left(x+\frac{3}{2}\right)\right) \\ &= \begin{cases} \frac{x}{2}+\frac{5}{4}, & -\frac{5}{2}\le x\le -\frac{3}{2}, \\ -1-x, & -\frac{3}{2}\le x\le -1, \\ \frac{x}{2}-\frac{1}{4}, & \frac{1}{2}\le x\le \frac{3}{2}, \\ 2-x, & \frac{3}{2}\le x\le 2, \\ 0, \quad x<-\frac{5}{2}, & -1<x<\frac{1}{2}, \text{ and } x>2 \end{cases} \end{aligned}$$

The solution $u\,(x,t_0)$ is plotted at times $t_0 = 0, 1/2, 1, 3/2$ in Figure 2. A 3D version of $u\,(x,t)$ is plotted in Figure 3.

(ii) Repeat the procedure in (i) for a string that has zero initial displacement but is given an initial velocity

$$u_t\,(x,0) = g\,(x) = \begin{cases} -1, & -1\le x<0 \\ 1, & 0\le x\le 1 \\ 0, & x<-1 \text{ and } x>1 \end{cases}$$

**Solution:** D'Alembert's formula is

$$u\,(x,t) = \frac{1}{2}\left(f\,(x-t)+f\,(x+t)+\int_{x-t}^{x+t} g\,(s)\,ds\right)$$

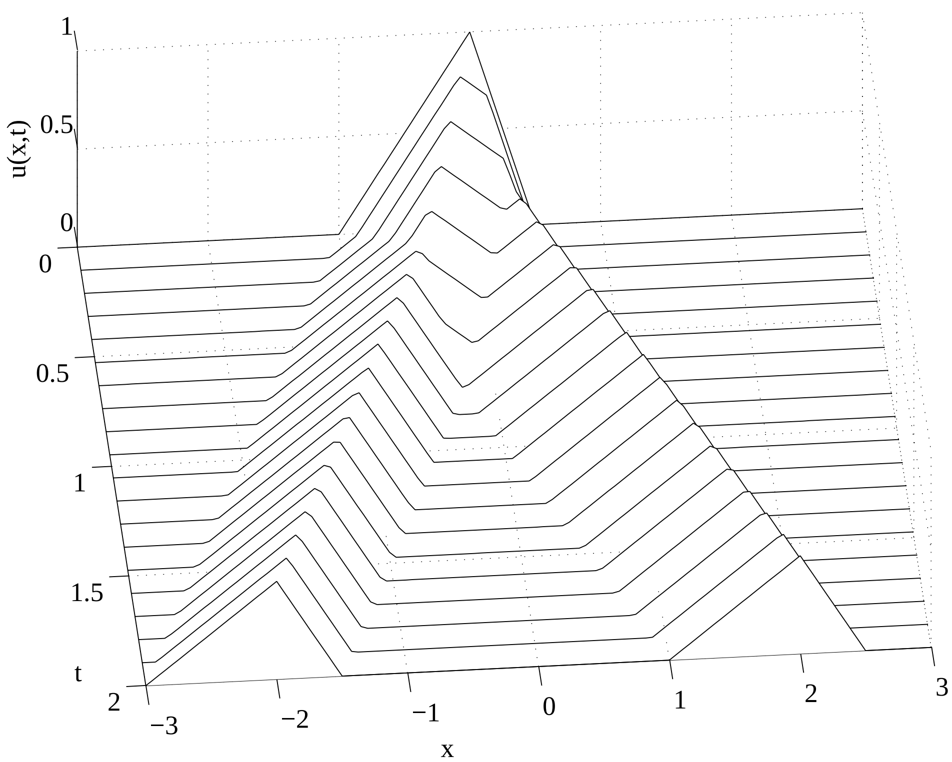


Figure 3: 3D version of $u(x,t)$ for 1(a).

In this case $f(s) = 0$ so that

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

The problem reduces to noting where $x \pm t$ lie in relation to $\pm 1$ and evaluating the integral. These characteristics are plotted in Figure 1 in the notes.

You can proceed in two ways. First, you can draw two more characterstics $x \pm t = 0$ so you can decide where the integration variable $s$ is with respect to zero, and hence if $g(s) = -1$ or 1. The second way is to note that for $a < b$ and $|a|, |b| < 1$,

$$\int_a^b g(s)\,ds = |b| - |a|$$

for positive and negative $a, b$. I'll use the second method; the answers you get from the first are the same.

In Region $R_1$,

$$|x \pm t| \le 1$$

and hence there are 3 cases: $x-t<0$, $x$

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ &= \frac{1}{2}\left(|x+t|-|x-t|\right) \end{aligned}$$

In Region $R_2$, $x+t>1$ and $-1<x-t<1$, so that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\int_{x-t}^{1}+\int_{1}^{x+t}\right) g(s)\,ds = \frac{1}{2}\int_{x-t}^{1} g(s)\,ds \\ &= \frac{1}{2}\left(1-|x-t|\right) \end{aligned}$$

In Region $R_3$, $x-t<-1$ and $-1<x+t<1$, so that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\int_{x-t}^{-1}+\int_{-1}^{x+t}\right) g(s)\,ds = \frac{1}{2}\int_{-1}^{x+t} g(s)\,ds = \frac{1}{2}\left(|x+t|-|-1|\right) \\ &= \frac{1}{2}\left(|x+t|-1\right) \end{aligned}$$

In Region $R_4$, $x+t>1$ and $x-t<-1$, so that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\int_{x-t}^{-1}+\int_{-1}^{1}+\int_{1}^{x+t}\right) g(s)\,ds \\ &= \frac{1}{2}\int_{-1}^{1} g(s)\,ds = \frac{1}{2}(-1+1) \\ &= 0 \end{aligned}$$

In Region $R_5$, $x+t<-1$ and hence $u(x,t)=0$. In region $R_6$, $x-t>1$, so that $u(x,t)=0$.

At $t=0$,

$$u(x,0)=\frac{1}{2}\int_{x}^{x} g(s)\,ds = 0$$

At $t=1/2$, the regions $R_n$ are given in the notes and

$$u\left(x,\frac{1}{2}\right)=\begin{cases} \frac{1}{2}\left(\left|x+\frac{1}{2}\right|-\left|x-\frac{1}{2}\right|\right), & x\in R_1=\left[-\frac{1}{2},\frac{1}{2}\right] \\ \frac{1}{2}\left(1-\left|x-\frac{1}{2}\right|\right), & x\in R_2=\left[\frac{1}{2},\frac{3}{2}\right] \\ \frac{1}{2}\left(\left|x+\frac{1}{2}\right|-1\right), & x\in R_3=\left[-\frac{3}{2},-\frac{1}{2}\right] \\ 0, & x\in R_5,R_6=\{|x|>3/2\} \end{cases}$$

The absolute values are easy to resolve (i.e. write without them) in this case. For example, for $x\in[-1/2,1/2]$, we have $|x-1/2|=-(x-1/2)$. Thus,

$$u\left(x,\frac{1}{2}\right)=\begin{cases} x, & x\in R_1=\left[-\frac{1}{2},\frac{1}{2}\right] \\ \frac{3}{4}-\frac{x}{2}, & x\in R_2=\left[\frac{1}{2},\frac{3}{2}\right] \\ -\frac{3}{4}-\frac{x}{2}, & x\in R_3=\left[-\frac{3}{2},-\frac{1}{2}\right] \\ 0, & x\in R_5,R_6=\{|x|>3/2\} \end{cases}$$

At $t = 1$, the regions $R_n$ are given in the notes and

$$u(x,1) = \begin{cases} \frac{1}{2}(1 - |x-1|), & x \in R_2 = [0,2], \\ \frac{1}{2}(|x+1| - 1), & x \in R_3 = [-2,0], \\ 0, & x \in R_5, R_6 = \{|x| > 3/2\}. \end{cases}$$

You could leave your answer like this, or write it without absolute values (have to divide $[0,2]$ and $[-2,0]$ into cases):

$$u(x,1) = \begin{cases} x/2, & x \in [0,1], \\ \frac{1}{2}(2-x), & x \in [1,2], \\ -\frac{1}{2}(x+2) & x = [-2,-1] \\ x/2, & x = [-1,0], \\ 0, & x \in R_5, R_6 = \{|x| > 3/2\}. \end{cases}$$

At $t = 3/2$, the regions $R_n$ are not given explicitly, but can be found from Figure 1 in the notes by nothing where the line $t = 3/2$ crosses each region:

$$u\left(x, \frac{3}{2}\right) = \begin{cases} \frac{1}{2}\left(1 - \left|x - \frac{3}{2}\right|\right), & x \in R_2 = \left[\frac{1}{2}, \frac{5}{2}\right] \\ \frac{1}{2}\left(\left|x + \frac{3}{2}\right| - 1\right), & x \in R_3 = \left[-\frac{5}{2}, -\frac{1}{2}\right] \\ 0, & x \in R_4, R_5, R_6 = \{|x| > 5/2 \text{ or } |x| < 1/2\} \end{cases}$$

Again, you could leave your answer like this, or write it without absolute values (have to divide $[1/2, 5/2]$ and $[-5/2, -1/2]$ into cases):

$$u\left(x, \frac{3}{2}\right) = \begin{cases} \frac{1}{2}\left(x - \frac{1}{2}\right), & x \in R_2 = \left[\frac{1}{2}, \frac{3}{2}\right] \\ \frac{1}{2}\left(\frac{5}{2} - x\right), & x \in R_2 = \left[\frac{3}{2}, \frac{5}{2}\right] \\ -\frac{1}{2}\left(x + \frac{5}{2}\right), & x \in R_3 = \left[-\frac{5}{2}, -\frac{3}{2}\right] \\ \frac{1}{2}\left(x + \frac{1}{2}\right), & x \in R_3 = \left[-\frac{3}{2}, -\frac{1}{2}\right] \\ 0, & x \in R_4, R_5, R_6 = \{|x| > 5/2 \text{ or } |x| < 1/2\} \end{cases}$$

The solution $u(x, t_0)$ is plotted at times $t_0 = 0, 1/2, 1, 3/2$ in Figure 4.

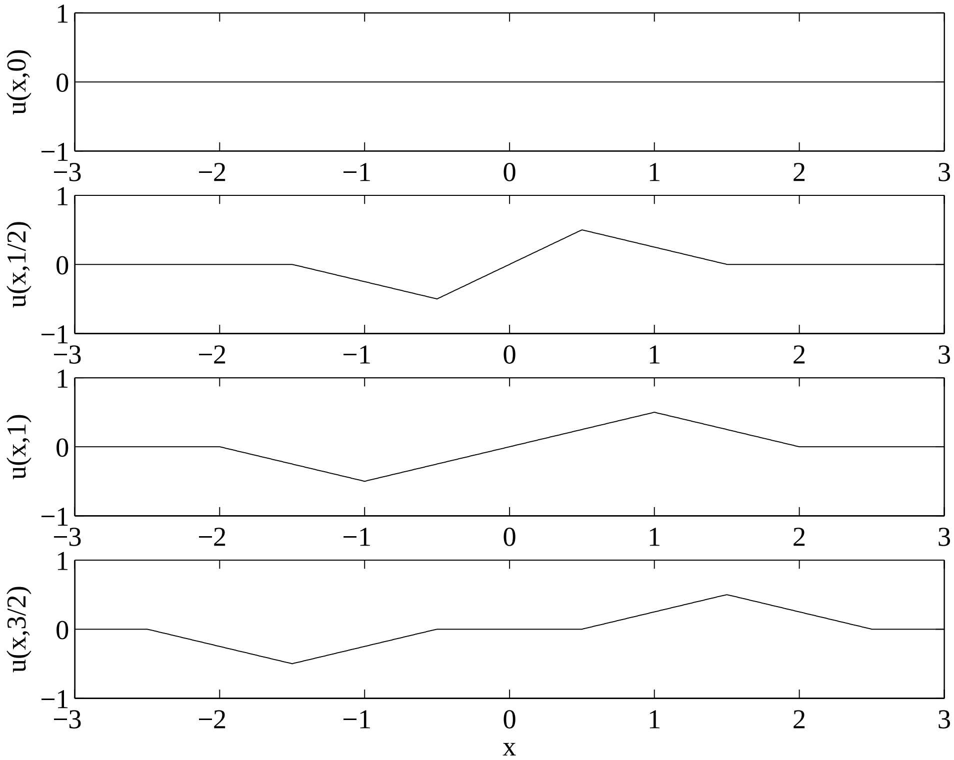


Figure 4: Plot of $u(x, t_0)$ for $t_0 = 0, 1/2, 1, 3/2$ for 1(b).

# 4 Question 1

[20 points total]

Suppose you shake the end of a rope of dimensionless length 1 at a certain frequency $\omega$. The opposite end of the rope is fixed to a wall. We aim to find the special frequencies $\omega$ at which certain points along the rope remain fixed in mid air. We negelect gravity and friction and model the waves on the rope using the 1D wave equation:

$$u_{tt} = u_{xx}, \qquad 0 < x < 1, \qquad t > 0 \tag{1}$$

where $u$ is the displacement of the rope away from it's rest state. The end at the wall (at $x = 0$) is fixed:

$$u(0,t) = 0, \qquad t > 0. \tag{2}$$

You shake the other end (at $x = 1$) sinusoidally, with frequency $\omega$, and give it the displacement

$$u(1,t) = \sin \omega t, \qquad t > 0. \tag{3}$$

We assume the rope has zero initial position and velocity

$$u(x,0) = 0, \qquad 0 < x < 1, \tag{4}$$

$$u_t(x,0) = 0, \qquad 0 < x < 1. \tag{5}$$

(a) [10 points] Find a solution of the form

$$u_S(x,t) = X(x) \sin \omega t \tag{6}$$

that satisfies the PDE (1) and the BCs (2) and (3). (Don't worry about the ICs yet.) Where is the rope stationary (i.e. $u_s(x,t) = 0$)? For what values of $\omega$ is your solution invalid?

**Solution:** From (6), we have

$$\begin{aligned} u_{Sxx} &= X''(x) \sin \omega t \\ u_{Stt} &= -\omega^2 X(x) \sin \omega t \end{aligned}$$

Thus the PDE (1) implies

$$X'' + \omega^2 X = 0$$

Solving gives

$$X(x) = a \cos(\omega x) + b \sin(\omega x) \tag{7}$$

Substituting (1) into the BC (2) gives

$$0 = X(0)\sin\omega t$$

and hence $X(0) = 0$. Thus (7) becomes

$$0 = X(0) = a$$

and hence

$$X(x) = b\sin\omega x$$

BC (3) is satisfied if

$$1 = X(1) = b\sin\omega$$

Thus $b = 1/\sin\omega$ and

$$u_S(x,t) = \frac{\sin\omega x}{\sin\omega}\sin\omega t$$

Rope stationary at $\omega x = n\pi$,

$$x = \frac{n\pi}{\omega}, \qquad n = 1, 2, 3...$$

such that $n\pi/\omega < 1$. Solution invalid when $\omega = n\pi$ for some $n = 1, 2, 3, ....$

(b) [10 points] Use $u_S(x,t)$ from part (a) to find the full solution $u(x,t)$ to the PDE (1), the BCs (2) and (3), and the ICs (4) and (5). Hint: use $u_S(x,t)$ and $u(x,t)$ to find a wave problem for some quantity $v(x,t)$ that has Type I BCs (i.e. zero displacement) at $x = 0$, 1. Then write down D'Alembert's solution (without derivation) to satisfy the PDE and initial conditions for this problem (don't evalutate the integral in D'Alembert's solution). Then adjust D'Alembert's solution to handle the BCs at $x = 0, 1$. You don't need to evaluate the integral.

**Solution:** Let $v = u - u_S$. Then

$$v_{tt} = v_{xx}, \qquad 0 < x < 1, \qquad t > 0$$

$$v(0,t) = 0 = v(1,t), \qquad t > 0$$

$$\begin{aligned} v(x,0) &= u(x,0) - u_S(x,0) = 0 \\ v_t(x,0) &= u_t(x,0) - u_{St}(x,0) = 0 - \frac{\omega}{\sin\omega}\sin\omega x \end{aligned}$$

D'Alembert's solution for the infinite string is

$$v(x,t) = -\frac{\omega}{2\sin\omega}\int_{x-t}^{x+t}\sin(\omega s)\,ds$$

To satisfy the BCs, we must extend $\sin(\omega s)$ to an odd periodic function in $s$. Since $\sin(\omega s)$ is already odd, we merely need it's 2−periodic extension,

$$\hat{f}(s) = \sin(\omega(s \bmod 2))$$

where we assume $(-s) \bmod 2 = -(s \bmod 2)$, so that

$$v(x,t) = -\frac{\omega}{2\sin\omega}\int_{x-t}^{x+t}\hat{f}(s)\,ds$$

Thus

$$\begin{aligned} u(x,t) &= v(x,t) + u_S(x,t) \\ &= -\frac{\omega}{2\sin\omega}\int_{x-t}^{x+t}\sin(\omega(s \bmod 2))\,ds \\ &\quad +\frac{\sin\omega x}{\sin\omega}\sin\omega t \end{aligned}$$

# 5 Question 2

[30 points total]

Consider the following quasi-linear PDE,

$$\frac{\partial u}{\partial t} + (1+2u)\frac{\partial u}{\partial x} = -u; \qquad u(x,0) = f(x) \tag{8}$$

where the initial condition is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2-|x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2+x, & -1 \le x \le 0 \\ 2-x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(a) [8 points] Find the parametric solution, using $r$ as your parameter along a characteristic and $s$ to label the characteristic (i.e. the initial value of $x$). First write down the relevant ODEs for $\partial t/\partial r$, $\partial x/\partial r$, $\partial u/\partial r$. Take the initial conditions $t = 0$ and $x = s$ at $r = 0$. Using the initial condition in (8), write down the IC for $u$ at $r = 0$, in terms of $s$. Solve for $t$, $u$ and $x$ (in that order!) as functions of $r$, $s$. When integrating for $x$, be careful: $u$ depends on $r$!

**Solution:** The parametric ODEs are

$$\frac{\partial t}{\partial r} = 1, \qquad \frac{\partial x}{\partial r} = 1+2u, \qquad \frac{\partial u}{\partial r} = -u.$$

The ICs are

$$t(0;s) = 0, \qquad x(0;s) = s, \qquad u(0;s) = f(x(0;s)) = f(s) \tag{9}$$

Solving for $t$ and $u$ gives

$$t = r + c_1, \qquad \ln u = -r + c_2$$

where $c_{1,2}$ are constants of integration. Thus

$$u = c_3 e^{-r}$$

Imposing the ICs (9) gives

$$t = r, \qquad u = f(s)\, e^{-r} = f(s)\, e^{-t}$$

Substituting $u$ into the equation for $x$ gives

$$\frac{\partial x}{\partial r} = 1 + 2f(s)\, e^{-r}$$

and integrating yields

$$x = r - 2f(s)\, e^{-r} + c_4$$

Imposing the IC (9) gives

$$\begin{aligned} x &= r - 2f(s)\left(e^{-r} - 1\right) + s \\ &= t - 2f(s)\left(e^{-t} - 1\right) + s \end{aligned}$$

(b) [8 points] At what time $t_{sh}$ and position $x_{sh}$ does your parametric solution break down? Hint: you might need to consider negative values of $f'(s)$.

**Solution:** The solution breaks down when the Jacobian is zero:

$$\begin{aligned} J &= \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r} = -\frac{\partial x}{\partial s} \\ &= -\left(-2f'(s)\left(e^{-r} - 1\right) + 1\right) \\ &= 0 \end{aligned}$$

Since $r = t$, we solve for $t$ to obtain

$$t_{sh} = -\ln\left(1 + \frac{1}{2f'(s)}\right)$$

Note that $f'(s)$ is either 0, yielding infinite $t_{sh}$, 1, yielding negative $t_{sh}$, and $-1$, yielding

$$t_{sh} = -\ln\left(\frac{1}{2}\right) = \ln 2$$

Thus the shock time is at $\ln 2$. An $s$ value where $f'(s) = -1$ is $s = 1$, so that the shock location is

$$\begin{aligned} x_{sh} &= t_{sh} - 2f(1)\left(e^{-t_{sh}} - 1\right) + 1 \\ &= \ln 2 - 2\left(\frac{1}{2} - 1\right) + 1 = \ln 2 + 2 \end{aligned}$$

(c) [4 points] Write down $x$ in terms of $t$, $s$ and $f(s)$. For each of $s = -1, 0, 1$, write down $x$ as a function of $t$.

**Solution:**

$$\begin{aligned} s &= -1 : x = t - 2\left(e^{-t} - 1\right) - 1 \\ s &= 0 : x = t - 4\left(e^{-t} - 1\right) \\ s &= 1 : x = t - 2\left(e^{-t} - 1\right) + 1 \end{aligned}$$

(d) [4 points] Fill in the table below, using your result from either question (a) or (c) to obtain $u$ and $x$ at the $s-$values listed at time $t = \ln 2$ (note that $e^{-\ln 2} = \frac{1}{2}$, and you may use $\ln 2 = 0.7$):

$t = \ln 2$

| $s =$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $u =$ | | | |
| $x =$ | | | |

**Solution:**

$t = \ln 2$

| $s =$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $u =$ | $1/2$ | 1 | $1/2$ |
| $x =$ | $\ln 2$ | $\ln 2 + 2$ | $\ln 2 + 2$ |

(e) [6 points] Plot the initial $f(x)$ vs. $x$. Then, on the SAME plot, plot $u(x,t)$ at $t = \ln 2$ by plotting the three points $(x, u)$ from the table in part (d).