Test 2
Introduction to PDE
MATH 3363-25820 (Fall 2009)

Solutions to Test 2

This exam has 2 questions, for a total of 20 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Upon finishing PLEASE write and sign your pledge below:

On my honor I have neither given nor received any aid on this exam.

____

1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided. Be sure to show a few key intermediate steps when deriving results - answers only will not get full marks.

2 Given

The solution to the 1d wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

subject to the ICs

$$\begin{aligned} u(x,0) &= f(x), \quad -\infty < x < \infty, \\ u_t(x,0) &= g(x), \quad -\infty < x < \infty, \end{aligned}$$

is

$$u(x,t) = F(x-ct) + G(x+ct)$$

where

$$\begin{aligned} G(x) &= \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds \\ F(x) &= \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds \end{aligned}$$

3 Questions

8 points 1. Suppose that an "infinite string" is initially streched into the shape of a single rectangular pulse and is let go from the rest. We model the problem using the 1D wave equation

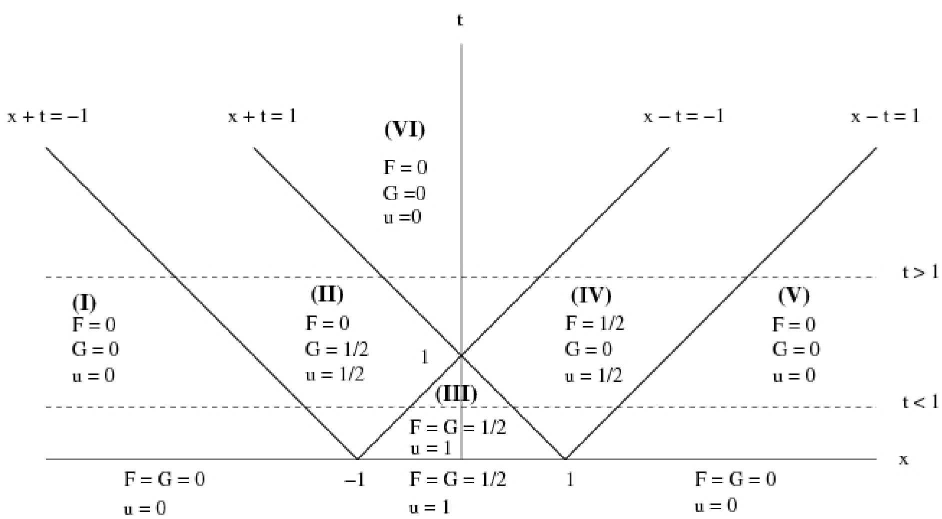
$$u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

subject to the initial conditions

$$u(x,0) = f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} = \begin{cases} 0, & x < -1 \\ 1, & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

$$u_t(x,0) = 0.$$

(4) (a) Plot the four characteristics: $x - t = -1$, $x + t = -1$, $x - t = 1$ and $x + t = 1$ in the space-time plane (xt). Show that these four characteristics divide the space-time plane (xt) into six distinct regions. In each region, show that the solution u is constant, and give its value.



Solutoin: Since the string is initially at rest $u_t(x,0) = g(x) = 0$, then, using the given formula,

+1 $$F(x) = G(x) = \frac{1}{2} f(x).$$

Thus

+1 $$u(x,t) = F(x-t) + G(x+t) = \frac{1}{2}\big[f(x-t) + f(x+t)\big].$$

The initial condition (rectangular pulse) $u(x,0) = f(x)$ splits into two parts; half moves to the left and half to the right, each traveling at speed $c = 1$.
The characteristics $x \pm t = -1$ and $x \pm t = 1$ are plotted in the figure above. They divide the space-time plane (xt) into six distinct regions I, II, III, IV, V, and VI, where the values of the solution u are constant:

+2 $u = 0$ in region I, $\quad u = 1/2$ in region II, $\quad u = 1$ in region III,

$u = 1/2$ in region IV, $\quad u = 0$ in region V, $\quad u = 0$ in region VI.

(2) (b) Determine analytic formulas for $u(x,t)$ for $t < 1$ and $t > 1$.

Solutoin: Using characteristics as sketched in the figure above, there are two distinct regions $t < 1$ and $t > 1$. In each, the solution has five different form, depending on x.

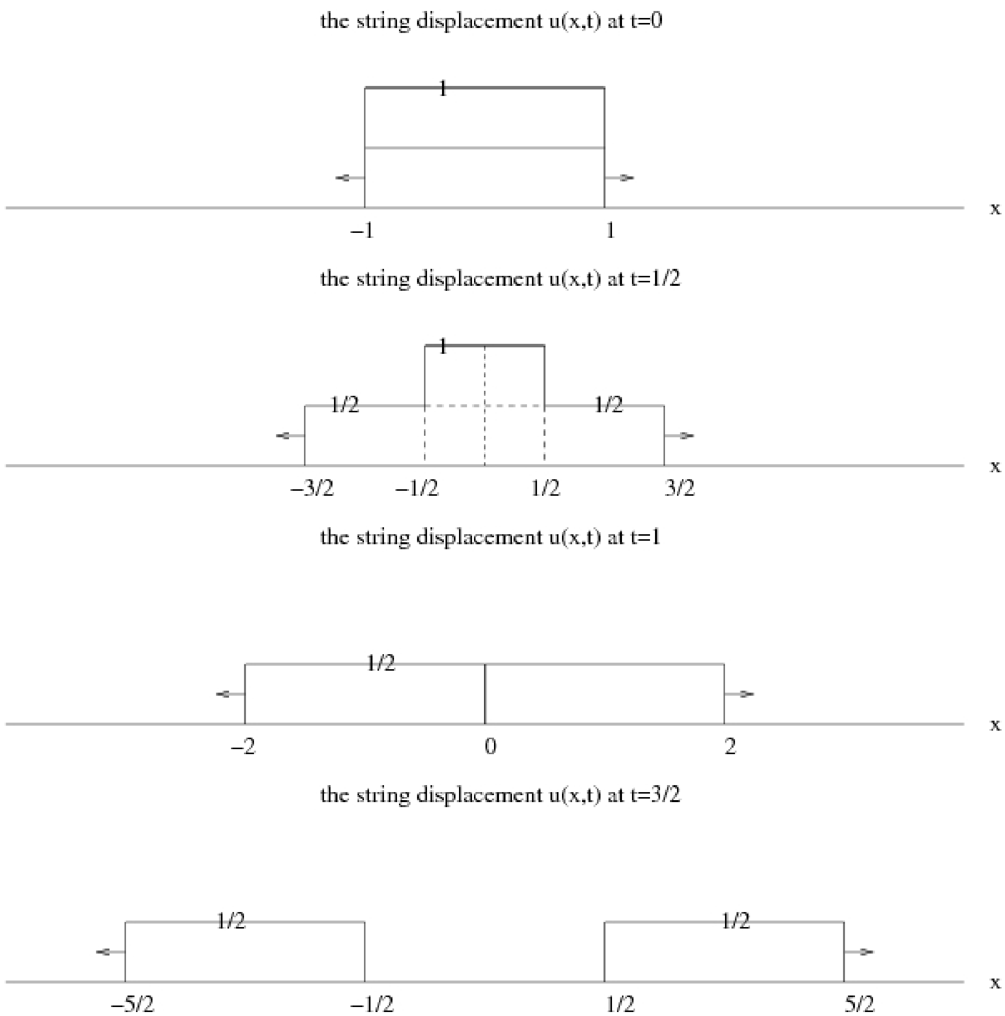
For a fixed $t < 1$, the t-valued horizontal line, as sketched in the figure above, is divided by the regions I, II, III, IV, V into five sub-intervals $(-\infty, -1-t)$, $(-1-t, -1+t)$, $(-1+t, 1-t)$, $(1-t, 1+t)$, $(1+t, \infty)$, where the values of the solution u are constant: 0, 1/2, 1, 1/2, 0, i.e.,

+1
$$\text{For } t < 1, \quad u(x,t) = \begin{cases} 0 & -\infty < x < -1-t \\ 1/2 & -1-t < x < -1+t \\ 1 & -1+t < x < 1-t \\ 1/2 & 1-t < x < 1+t \\ 0 & 1+t < x < \infty \end{cases}$$

For a fixed $t > 1$, the t-valued horizontal line, as sketched in the figure above, is divided by the regions I, II, VI, IV, V into five sub-intervals $(-\infty, -1-t)$, $(-1-t, 1-t)$, $(1-t, -1+t)$, $(-1+t, 1+t)$, $(1+t, \infty)$, where the values of the solution u are constant: 0, 1/2, 0, 1/2, 0, i.e.,

+1
$$\text{For } t > 1, \quad u(x,t) = \begin{cases} 0 & -\infty < x < -1-t \\ 1/2 & -1-t < x < 1-t \\ 0 & 1-t < x < -1+t \\ 1/2 & -1+t < x < 1+t \\ 0 & 1+t < x < \infty \end{cases}$$

(2) (c) Illustrate the time evolution of the string by sketching the ux-profiles $u(x,t)$ of the string displacement for $t = 0,\ 1/2,\ 1,\ 3/2$.



Solutoin: The solution $u(x,t)$ is given by adding together the two rectangular pulses $\frac{1}{2}f(x-t)$ and $\frac{1}{2}f(x+t)$. The pusles overlap until the left end of the right-moving one passes the right end of the other. Since each is traveling at speed $c = 1$,they are moving apart at velocity 2. The ends are initially a distance 2 apart, and hence the time at which the two pulses separate is

+1
$$t = \frac{\text{distance}}{\text{velocity}} = \frac{2}{2} = 1$$

+1 The figure above illustrates the time evolution of the string by sketching the ux-profiles $u(x,t)$ of the string displacement for $t = 0,\ 1/2,\ 1,\ 3/2$.

12 points 2. Consider the quasi-linear PDE

$$\frac{\partial u}{\partial t} + (1+u)\frac{\partial u}{\partial x} = 0, \quad u(x,0) = f(x)$$

where the initial condition $f(x)$ is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2 - |x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2 + x, & -1 \le x \le 0 \\ 2 - x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(4) (a) By writing the PDE in the form $(A, B, C) \cdot (\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1) = 0$, find the parametric solution using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x). First write down the relevant ODEs for $\frac{dt}{dr}$, $\frac{dx}{dr}$, $\frac{du}{dr}$. Take the initial conditions $t = 0$ and $x = s$ at $r = 0$. Using the initial condition, write down the IC for u at $r = 0$, in terms of s. Solve for t, u and x as functions of r, s.

Solutoin: To find the parametric solution, we can write the PDE as

+1 $$(1, 1+u, 0) \cdot \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1\right) = 0.$$

Thus the parametric solution $(t(r;s), x(r;s), u(r;s))$ is defined, for a fixed parameter s, by the ODEs

+1 $$\frac{dt}{dr} = 1, \quad \frac{dx}{dr} = 1 + u, \quad \frac{du}{dr} = 0$$

with initial conditions at $r = 0$,

+1 $$t = 0, \quad x = s, \quad u = u(x,0) = u(s,0) = f(s).$$

Integrating the ODEs and imposing the ICs gives

+1 $$\begin{aligned} t(r;s) &= r \\ u(r;s) &= f(s) \\ x(r;s) &= (1 + f(s))r + s = (1 + f(s))t + s \end{aligned}$$

(3) (b) At what time t_{sh} and location x_{sh} does your parametric solution break down?

Solutoin: To find the time t_{sh} and position x_{sh} when and where a shock first forms, we find the Jacobian:

+0.5
$$J = \frac{\partial(x,t)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = x_r t_s - x_s t_r = -x_s = -(f'(s)t + 1)$$

Shocks occur (the solution breaks down) where $J = 0$, i.e. where

+0.5
$$t = -\frac{1}{f'(s)}$$

The first shock occurs at

$$t_{\text{sh}} = \min\left(-\frac{1}{f'(s)}\right) = -\max\left(\frac{1}{f'(s)}\right) = -\frac{1}{\min(f'(s))}$$

Since

+1
$$f'(x) = \begin{cases} 0, & x < -1 \\ 1, & -1 < x < 0 \\ -1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

thus $\min(f'(s)) = -1$, we have

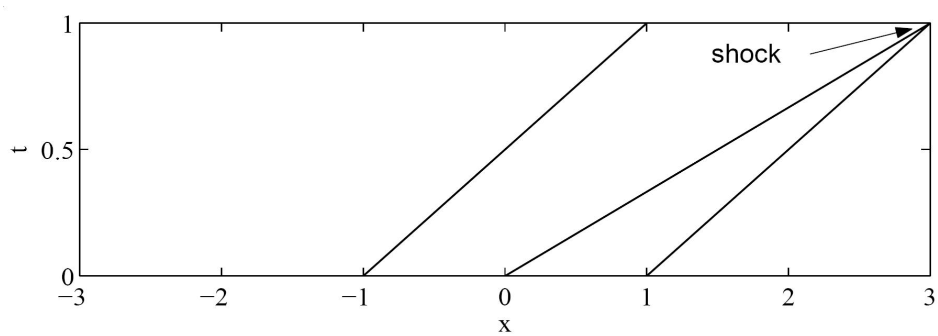
+0.5
$$t_{\text{sh}} = -\frac{1}{\min(f'(s))} = 1.$$

Any of the characteristics where $f'(s) = \min(f'(s)) = -1$ can be used to find the location of the shock at $t_{\text{sh}} = 1$. For e.g., with $s = 1/2$, the location of the shock at $t_{\text{sh}} = 1$ is

+0.5
$$x_{\text{sh}} = \left(1 + f\left(\frac{1}{2}\right)\right)1 + \frac{1}{2} = \left(1 + \left(2 - \frac{1}{2}\right)\right)1 + \frac{1}{2} = 3.$$

Any other value of $s \in (0,1)$ where $f'(s) = \min(f'(s)) = -1$ will give the same x_{sh}.

(2) (c) For each of $s = -1, 0, 1$, write down the characteristic x as a functions of t and plot the characteristic in the space-time plane (xt).



Solutoin: The $s = -1, 0, 1$ characteristics are given by

$$s = -1: \quad x = (1 + f(-1))t - 1 = 2t - 1$$

+2
$$s = 0: \quad x = (1 + f(0))t + 0 = 3t$$

$$s = 1: \quad x = (1 + f(1))t + 1 = 2t + 1$$

These are plotted in the figure above.

(2) (d) The tables below are useful as a plotting aid for (e). Fill in the tables using your result from (a) or (c) to obtain u and x at the s-values listed at time $t = 1/2, 1$.

$t = \frac{1}{2}$

$s =$	-1	0	1
$u =$			
$x =$			

$t = 1$

$s =$	-1	0	1
$u =$			
$x =$			

Solutoin:

+1

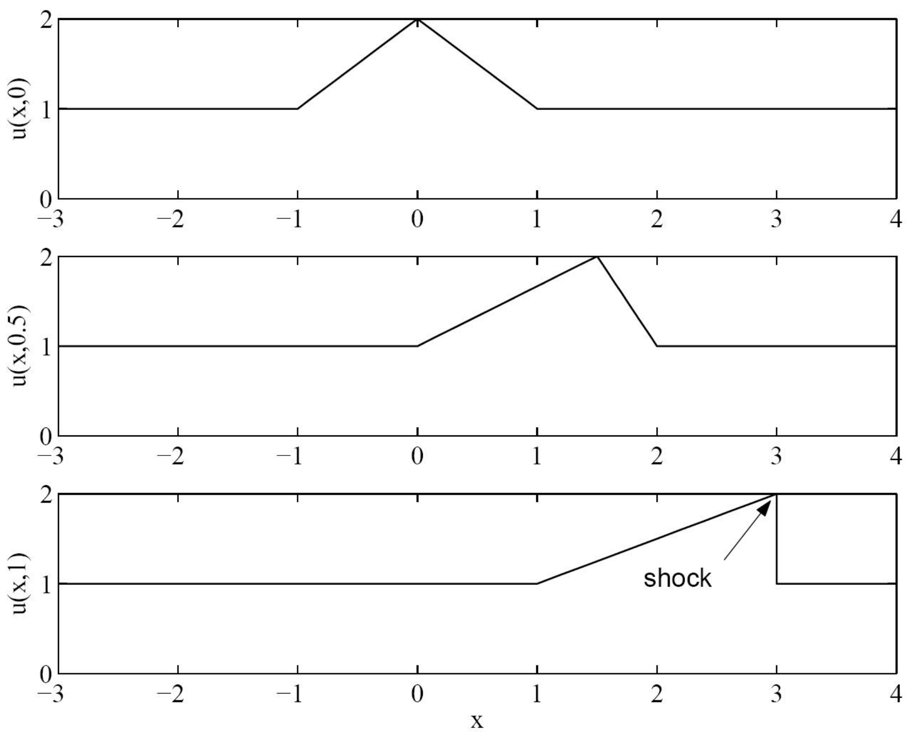
$t = \frac{1}{2}$

$s =$	-1	0	1
$u =$	1	2	1
$x =$	0	$3/2$	2

+1

$$t = 1 \qquad \begin{array}{rrrr} s = & -1 & 0 & 1 \\ u = & 1 & 2 & 1 \\ x = & 1 & 3 & 3 \end{array}$$

(1) (e) Illustrate the time evolution of the solution by sketching the ux-profiles $u(x,t)$ for $t = 0,\ 1/2,\ 1$.



Solutoin: A plot of $u(x, 1/2)$ is made by plotting the three points (x, u) from the table for $t = 1/2$ and connecting the dots (see middle plot in the figure above). Similarly, $u(x, t_{\rm sh}) = u(x, 1)$ is plotted in the last plot.

+1