Test 3
Introduction to PDE
MATH 3363-25820 (Fall 2009)

Name and UH-ID:

This exam has 3 questions, for a total of 20 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Upon finishing PLEASE write and sign your pledge below:
On my honor I have neither given nor received any aid on this exam.

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1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided. Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given in Section 2, without proof, on any question in the exam.

2 Given

You may use the following without proof:

The Laplacian ∇^2 in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

1D Sturm-Liouville Problems: The eigen-solution to

$$X'' + \lambda X = 0; \qquad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, ..$$

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The eigen-solution to

$$Y'' + \lambda Y = 0; \qquad Y'(0) = 0 = Y'(L)$$

is

$$Y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, ..$$

Orthogonality condition for sines and cosines: for any $L > 0$ (e.g. $L = 1$, π, $\pi/2$, etc)

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

The general solution to Bessel's Equation

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - m^2\right)R(r) = 0, \qquad m = 0, 1, 2, 3, ...$$

is

$$R_m(r) = c_{m1}J_m\left(\sqrt{\lambda}r\right) + c_{m2}Y_m\left(\sqrt{\lambda}r\right)$$

where c_{mn} are constants of integration, $J_m\left(\sqrt{\lambda}r\right)$ is bounded as $r \to 0$ and

$$\left|Y_m\left(\sqrt{\lambda}r\right)\right| \to \infty \text{ as } r \to 0.$$

Orthogonality for Bessel Functions J_n,

$$\int_0^1 rJ_n(j_{n,m}r)\,J_k(j_{k,l}r)\,dr = 0, \qquad \text{if } n \neq k \text{ or } m \neq l$$

where $j_{n,m}$ is the m'th zero of the Bessel function of order n. If $n = k$ and $m = l$, just write

$$\int_0^1 r\left(J_n(j_{n,m}r)\right)^2 dr \qquad (> 0)$$

A useful result derived from the Divergence Theorem,

$$\int\int_D v\nabla^2 v dV = -\int\int_D |\nabla v|^2 dV + \int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS \tag{1}$$

for any 2D or 3D region D with closed boundary ∂D.

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3 Questions

$\boxed{\text{6 points}}$ 1. Solve Laplace's Equation on the half unit disc,

$$\Delta u(r, \theta) = 0$$

with BCs

$$u(1, \theta) = g(\theta), \quad u(0, \theta) \text{ bounded }, \quad 0 < \theta < \pi,$$
$$\frac{\partial u}{\partial \theta}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}(r, \pi) = 0, \quad 0 < r < 1.$$

Be sure to use any relevant given information to save time.

$\boxed{\text{10 points}}$ 2. Solve the Heat Problem on the half unit disc

$$v_t = \Delta v, \quad 0 < r < 1, \quad 0 < \theta < \pi, \quad t > 0,$$

subject to inhomogeneous BCs

$$v(1, \theta, t) = g(\theta), \quad v(0, \theta, t) \text{ bounded }, \quad 0 < \theta < \pi, \quad t > 0,$$
$$\frac{\partial v}{\partial \theta}(r, 0, t) = 0, \quad \frac{\partial v}{\partial \theta}(r, \pi, t) = 0, \quad 0 < r < 1, \quad t > 0,$$

and initial condition

$$v(r, \theta, 0) = f(r, \theta), \quad 0 < r < 1, \quad 0 < \theta < \pi.$$

Your solution will have coefficients in terms of integrals involving $f(r, \theta)$.

$\boxed{\text{4 points}}$ 3. Prove the solution to 2 is unique.
Hint: You'll need to use a result derived from the Divergence Theorem (on the given page). You don't need to consider r, θ: denoting the region by D and using dV will work fine.

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1.

1 (cont.)

2.

2 (cont.)

3.