Practice Test 3

1 Rules [requires student signature!]

1. I will use only pencils, pens, erasers, and straight edges to complete this exam.
2. I will NOT use calculators, notes, books or other aides.

Signature: ____ Date: ____________.

Please hand in this question sheet with your solutions following the exam.

2 Note

Work on problems (and sub-parts) in any order; just be sure to label the question. Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given on the next two pages, without proof, on any question in the exam.

3 Given

You may use the following without proof:

The Laplacian ∇^2 in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

1D Sturm-Liouville Problems: The eigen-solution to

$$X'' + \lambda X = 0; \qquad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, ..$$

The eigen-solution to

$$Y'' + \lambda Y = 0; \qquad Y'(0) = 0 = Y'(L)$$

is

$$Y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, ..$$

Orthogonality condition for sines and cosines: for any $L > 0$ (e.g. $L = 1, \pi, \pi/2$, etc)

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

The general solution to Bessel's Equation

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - m^2\right)R(r) = 0, \qquad m = 0, 1, 2, 3, ...$$

is

$$R_m(r) = c_{m1}J_m\left(\sqrt{\lambda}r\right) + c_{m2}Y_m\left(\sqrt{\lambda}r\right)$$

where c_{mn} are constants of integration, $J_m\left(\sqrt{\lambda}r\right)$ is bounded as $r \to 0$ and

$$\left|Y_m\left(\sqrt{\lambda}r\right)\right| \to \infty \text{ as } r \to 0.$$

Orthogonality for Bessel Functions J_n,

$$\int_0^1 rJ_n(j_{n,m}r)\,J_k(j_{k,l}r)\,dr = 0, \qquad \text{if } n \neq k \text{ or } m \neq l$$

where $j_{n,m}$ is the m'th zero of the Bessel function of order n. If $n = k$ and $m = l$, just write

$$\int_0^1 r\left(J_n\left(j_{n,m}r\right)\right)^2 dr \qquad (> 0)$$

A useful result derived from the Divergence Theorem,

$$\int\int_D v\nabla^2 v dV = -\int\int_D |\nabla v|^2 \, dV + \int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS \tag{1}$$

for any 2D or 3D region D with closed boundary ∂D.

4 Questions

(a) [10 marks] Solve Laplace's Equation on the quarter unit disc,

$$\nabla^2 u(r,\theta) = 0$$

with BCs

$$u(1,\theta) = g(\theta), \qquad u(0,\theta) \text{ bounded}, \qquad 0 < \theta < \pi/2,$$
$$u(r,0) = 0, \qquad u\left(r, \frac{\pi}{2}\right) = 0, \qquad 0 < r < 1.$$

Be sure to use any relevant given information to save time.

(b) [12 marks] Solve the Heat Problem on the unit quarter disc

$$v_t = \nabla^2 v, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$

subject to inhomogeneous BCs

$$v(1,\theta,t) = g(\theta), \qquad v(0,\theta,t) \text{ bounded}, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$
$$v(r,0,t) = 0, \qquad v\left(r, \frac{\pi}{2}, t\right) = 0, \qquad 0 < r < 1, \qquad t > 0,$$

and initial condition

$$v(r,\theta,0) = f(r,\theta), \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2.$$

Your solution will have coefficients in terms of integrals involving $f(r,\theta)$.

(c) [8 marks] Prove the solution to (b) is unique. Hint: The steps follow those for the 1D rod, but you'll need to use a result derived from the Divergence Theorem (on the given page) instead of integration by parts. You don't need to consider r, θ: denoting the region by D and using dV will work fine.