Solutions to Practice Test 3

1 Rules [requires student signature!]

1. I will use only pencils, pens, erasers, and straight edges to complete this exam.
2. I will NOT use calculators, notes, books or other aides.

Signature: ____ Date: ____________.

Please hand in this question sheet with your solutions following the exam.

2 Note

Work on problems (and sub-parts) in any order; just be sure to label the question. Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given on the next two pages, without proof, on any question in the exam.

3 Given

You may use the following without proof:

The Laplacian ∇^2 in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

1D Sturm-Liouville Problems: The eigen-solution to

$$X'' + \lambda X = 0; \qquad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, ..$$

The eigen-solution to

$$Y'' + \lambda Y = 0; \qquad Y'(0) = 0 = Y'(L)$$

is

$$Y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, ..$$

Orthogonality condition for sines and cosines: for any $L > 0$ (e.g. $L = 1$, π, $\pi/2$, etc)

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

The general solution to Bessel's Equation

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - m^2\right)R(r) = 0, \qquad m = 0, 1, 2, 3, ...$$

is

$$R_m(r) = c_{m1}J_m\left(\sqrt{\lambda}r\right) + c_{m2}Y_m\left(\sqrt{\lambda}r\right)$$

where c_{mn} are constants of integration, $J_m\left(\sqrt{\lambda}r\right)$ is bounded as $r \to 0$ and

$$\left|Y_m\left(\sqrt{\lambda}r\right)\right| \to \infty \text{ as } r \to 0.$$

Orthogonality for Bessel Functions J_n,

$$\int_0^1 r J_n(j_{n,m}r)\, J_k(j_{k,l}r)\, dr = 0, \qquad \text{if } n \neq k \text{ or } m \neq l$$

where $j_{n,m}$ is the m'th zero of the Bessel function of order n. If $n = k$ and $m = l$, just write

$$\int_0^1 r \left(J_n\left(j_{n,m} r\right)\right)^2 dr \qquad (> 0)$$

A useful result derived from the Divergence Theorem,

$$\int\int_D v\nabla^2 v dV = -\int\int_D |\nabla v|^2 dV + \int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS \tag{1}$$

for any 2D or 3D region D with closed boundary ∂D.

4 Questions

(a) [10 marks] Solve Laplace's Equation on the quarter unit disc,

$$\nabla^2 u(r,\theta) = 0$$

with BCs

$$u(1,\theta) = g(\theta), \qquad u(0,\theta) \text{ bounded}, \qquad 0 < \theta < \pi/2,$$
$$u(r,0) = 0, \qquad u\left(r,\frac{\pi}{2}\right) = 0, \qquad 0 < r < 1.$$

Be sure to use any relevant given information to save time.

(b) [12 marks] Solve the Heat Problem on the unit quarter disc

$$v_t = \nabla^2 v, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$

subject to inhomogeneous BCs

$$v(1,\theta,t) = g(\theta), \qquad v(0,\theta,t) \text{ bounded}, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$
$$v(r,0,t) = 0, \qquad v\left(r,\frac{\pi}{2},t\right) = 0, \qquad 0 < r < 1, \qquad t > 0,$$

and initial condition

$$v(r,\theta,0) = f(r,\theta), \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2.$$

Your solution will have coefficients in terms of integrals involving $f(r,\theta)$.

(c) [8 marks] Prove the solution to (b) is unique. Hint: The steps follow those for the 1D rod, but you'll need to use a result derived from the Divergence Theorem (on the given page) instead of integration by parts. You don't need to consider r, θ: denoting the region by D and using dV will work fine.

Solution: (a) Separate variables as

$$u(r,\theta) = R(r)\,H(\theta)$$

so that the PDE becomes

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{H}\frac{d^2H}{d\theta^2} = \lambda$$

where λ is constant, since the l.h.s. depends only on r and the r.h.s. only on θ. Separating the BCs gives

$$\begin{aligned} u(r,0) &= 0 \Rightarrow H(0) = 0 \\ u\left(r,\frac{\pi}{2}\right) &= 0 \Rightarrow H(\pi/2) = 0 \end{aligned}$$

We solve for $H(\theta)$ first:

$$H'' + \lambda H = 0; \qquad H(0) = 0 = H(\pi/2).$$

Using the given information, we have

$$H(\theta) = \sin\left(\frac{n\pi\theta}{\pi/2}\right) = \sin(2n\theta), \qquad n = 1, 2, 3...$$

and $\lambda = 4n^2$. Thus, the equation for $R(r)$ is

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (2n)^2 R = 0$$

Try $R = r^\alpha$, so that $\alpha = \pm 2n$:

$$R = c_1 r^{2n} + c_2 r^{-2n}$$

For $R(0)$ to be bounded, we must have $c_2 = 0$. Putting things together gives

$$u_n(r,\theta) = R(r) H(\theta) = r^{2n}\sin(2n\theta), \qquad n = 1, 2, 3...$$

Using superposition, the general solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n u_n(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n}\sin(2n\theta)$$

where the A_n's are found using the BC at $r = 1$ and orthogonality,

$$g(\theta) = u(1,\theta) = \sum_{n=1}^{\infty} A_n \sin(2n\theta)$$

Multiplying by $\sin(2m\theta)$ and integrating from $\theta = 0$ to $\theta = \pi/2$ gives

$$\int_0^{\pi/2} g(\theta)\sin(2m\theta)\, d\theta = A_m \frac{\pi}{2}\frac{1}{2}$$

Thus

$$A_m = \frac{4}{\pi}\int_0^{\pi/2} g(\theta)\sin(2m\theta)\, d\theta$$

(b) First, let

$$V(r,\theta,t) = v(r,\theta,t) - u(r,\theta)$$

where $u(r,\theta)$ was found in part (a). Then $V(r,\theta,t)$ satisfies the homogeneous problem,

$$V_t = \nabla^2 V, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$

$$\begin{aligned} V(1,\theta,t) &= 0, \qquad V(0,\theta,t) \text{ bounded}, \qquad 0 < r < 1, \qquad t > 0, \\ V(r,0,t) &= 0, \qquad V\left(r,\frac{\pi}{2},t\right) = 0, \qquad 0 < \theta < \pi/2, \qquad t > 0, \end{aligned}$$

and initial condition

$$V(r,\theta,0) = f(r,\theta) - u(r,\theta), \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2.$$

We separate variables as

$$V(r,\theta,t) = R(r)\,H(\theta)\,T(t)$$

so that the PDE becomes

$$\frac{T'}{T} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2 H}\frac{d^2 H}{d\theta^2} = -\lambda$$

where λ is constant since the lhs depends only on t, and the rhs on r, θ. Since the solution must decay, we expect $\lambda > 0$. Or we could argue this from general theory. From the middle equation, we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{H}\frac{d^2 H}{d\theta^2} = \mu$$

again, since the lhs depends on r only and the rhs on θ only. Separating the BCs yields

$$V(1,\theta,t) = 0 \Rightarrow R(1) = 0$$

$$V(1,\theta,t) \text{ bounded } \Rightarrow |R(0)| < \infty$$

$$\begin{aligned} V(r,0,t) &= 0 \Rightarrow H(0) = 0, \\ V\left(r,\frac{\pi}{2},t\right) &= 0 \Rightarrow H\left(\frac{\pi}{2}\right) = 0. \end{aligned}$$

The problem for $H(\theta)$ is the same as before, thus

$$H(\theta) = \sin(2n\theta)$$

and $\mu = 4n^2$. Thus

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - (2n)^2\right)R = 0$$

and hence the general solution is

$$R(r) = c_1 J_{2n}\left(\sqrt{\lambda} r\right) + c_2 Y_{2n}\left(\sqrt{\lambda} r\right)$$

Since $R(0)$ must be bounded, $c_2 = 0$. The other BC is

$$1 = R(1) = c_1 J_{2n}\left(\sqrt{\lambda}\right)$$

Thus $\lambda_{nm} = j_{2n,m}^2$, where $j_{2n,m}$ is the m'th zero of J_{2n}.

Thus, for $n, m = 1, 2, 3, ...$

$$V_{nm}(r, \theta, t) = J_{2n}(r j_{2n,m}) \sin(2n\theta) e^{-\lambda_{nm} t}$$

solves the PDE and BCs. To solve the IC, we sum over n, m and use superposition,

$$V(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} J_{2n}(r j_{2n,m}) \sin(2n\theta) e^{-\lambda_{nm} t}$$

where the A_{nm}'s are found from the IC and orthogonality:

$$f(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} J_{2n}(r j_{2n,m}) \sin(2n\theta)$$

Multiplying by $r J_{2k}(r j_{2k,l}) \sin(2k\theta)$ and integrating in r, θ, we have

$$\int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (f(r, \theta) - u(r, \theta)) r J_{2k}(r j_{2k,l}) \sin(2k\theta) \, d\theta = A_{kl} \frac{\pi}{4} \int_{r=0}^{1} r (J_{2k}(r j_{2k,l}))^2 \, dr$$

Thus

$$A_{kl} = \frac{4}{\pi} \frac{\int_{r=0}^{1} \int_{\theta=0}^{\pi/2} (f(r, \theta) - u(r, \theta)) r J_{2k}(r j_{2k,l}) \sin(2k\theta) \, d\theta}{\int_{r=0}^{1} r (J_{2k}(r j_{2k,l}))^2 \, dr}$$

Finally,

$$v(r, \theta, t) = V(r, \theta, t) + u(r, \theta)$$

(c) Take 2 solutions v_1, v_2. Define the difference $h = v_1 - v_2$. Note that h satisfies

$$h_t = \nabla^2 h, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2, \qquad t > 0,$$

$$\begin{aligned} h(1, \theta, t) &= 0, \qquad h(0, \theta, t) \text{ bounded}, \qquad 0 < r < 1, \qquad t > 0, \\ h(r, 0, t) &= 0, \qquad h\left(r, \frac{\pi}{2}, t\right) = 0, \qquad 0 < \theta < \pi/2, \qquad t > 0, \end{aligned}$$

$$h(r, \theta, 0) = 0, \qquad 0 < r < 1, \qquad 0 < \theta < \pi/2.$$

Define the mean square difference between solutions,

$$\bar{V}(t) = \int\int_D h^2 dV \geq 0$$

Differentiate in time,

$$\begin{aligned} \frac{d\bar{V}(t)}{dt} &= \int\int_D 2hh_t dV = \int\int_D 2h\nabla^2 h dV \\ &= -2\int\int_D |\nabla v|^2 \, dV + 2\int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS \end{aligned}$$

But $v = 0$ on the boundary, so that

$$\frac{d\bar{V}(t)}{dt} = -2\int\int_D |\nabla v|^2 \, dV \leq 0$$

Note that at $t = 0$,

$$\bar{V}(0) = \int\int_D (f(r,\theta) - f(r,\theta))^2 \, dV = 0$$

Thus, $\bar{V}(t)$ is a non-negative, non-increasing function, that starts at zero. Hence $\bar{V}(t) = 0$ for all time, which implies by continuity that $h(r,\theta,t) = 0$ for all r, θ, t. Hence $v_1 = v_2$, and the solution to (b) is unique.