Test 3
Introduction to PDE
MATH 3363-25820 (Fall 2009)

Solutions to Test 3

This exam has 3 questions, for a total of 20 points.
Please answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.

Upon finishing PLEASE write and sign your pledge below:

On my honor I have neither given nor received any aid on this exam.

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1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided. Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks. You are free to use any of the information given in Section 2, without proof, on any question in the exam.

2 Given

You may use the following without proof:

The Laplacian ∇^2 in polar coordinates is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

1D Sturm-Liouville Problems: The eigen-solution to

$$X'' + \lambda X = 0; \qquad X(0) = 0 = X(L)$$

is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, ..$$

The eigen-solution to

$$Y'' + \lambda Y = 0; \qquad Y'(0) = 0 = Y'(L)$$

is

$$Y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 0, 1, 2, 3, ..$$

Orthogonality condition for sines and cosines: for any $L > 0$ (e.g. $L = 1$, π, $\pi/2$, etc)

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} L/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

The general solution to Bessel's Equation

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda r^2 - m^2\right)R(r) = 0, \qquad m = 0, 1, 2, 3, ...$$

is

$$R_m(r) = c_{m1}J_m\left(\sqrt{\lambda}r\right) + c_{m2}Y_m\left(\sqrt{\lambda}r\right)$$

where c_{mn} are constants of integration, $J_m\left(\sqrt{\lambda}r\right)$ is bounded as $r \to 0$ and

$$\left|Y_m\left(\sqrt{\lambda}r\right)\right| \to \infty \text{ as } r \to 0.$$

Orthogonality for Bessel Functions J_n,

$$\int_0^1 rJ_n\left(j_{n,m}r\right)J_k\left(j_{k,l}r\right)dr = 0, \qquad \text{if } n \neq k \text{ or } m \neq l$$

where $j_{n,m}$ is the m'th zero of the Bessel function of order n. If $n = k$ and $m = l$, just write

$$\int_0^1 r\left(J_n\left(j_{n,m}r\right)\right)^2 dr \qquad (> 0)$$

A useful result derived from the Divergence Theorem,

$$\int\int_D v\nabla^2 v dV = -\int\int_D |\nabla v|^2 dV + \int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS \tag{1}$$

for any 2D or 3D region D with closed boundary ∂D.

3 Questions

$\boxed{\text{6 points}}$ 1. Solve Laplace's Equation on the half unit disc,

$$\Delta u(r,\theta) = 0$$

with BCs

$$u(1,\theta) = g(\theta), \quad u(0,\theta) \text{ bounded }, \quad 0 < \theta < \pi,$$
$$\frac{\partial u}{\partial \theta}(r,0) = 0, \quad \frac{\partial u}{\partial \theta}(r,\pi) = 0, \quad 0 < r < 1.$$

Be sure to use any relevant given information to save time.

Solutoin: Separate variables as

0.5 points $$u(r,\theta) = R(r)H(\theta)$$

so that the PDE becomes

1 point $$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{H}\frac{d^2H}{d\theta^2} = \lambda$$

where λ is constant, since the l.h.s. depends only on r and the r.h.s. only on θ.

Separating the BCs gives

0.25 points $$u(0,\theta) \text{ bounded} \quad \Rightarrow \quad R(0) \text{ bounded}$$

0.25 points $$\frac{\partial u}{\partial \theta}(r,0) = 0 \quad \Rightarrow \quad H'(0) = 0$$

0.25 points $$\frac{\partial u}{\partial \theta}(r,\pi) = 0 \quad \Rightarrow \quad H'(\pi) = 0$$

We solve for $H(\theta)$ first:

0.5 points $$H'' + \lambda H = 0; \quad H'(0) = 0 = H'(\pi).$$

Using the given information, we have

0.5 points $$H(\theta) = \cos\left(\frac{n\pi\theta}{\pi}\right) = \cos(n\theta), \quad \lambda = (n)^2 = n^2, \quad n = 0, 1, 2, 3, \cdots.$$

Thus, the equation for $R(r)$ is

0.5 points $$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 R = 0,$$

and the general solution is

$$R(r) = \begin{cases} c_1 r^n + c_2 r^{-n}, & \text{for } n \neq 0 \\ \bar{c}_1 + \bar{c}_2 \ln r, & \text{for } n = 0 \end{cases}$$

For $R(0)$ to be bounded, we must have $c_2 = 0$ and $\bar{c}_2 = 0$. Putting things together gives

0.5 points $$u_n(r,\theta) = R(r)H(\theta) = r^n \cos(n\theta), \quad n = 0, 1, 2, 3, \cdots .$$

Using superposition, the general solution is

0.5 points $$u(r,\theta) = \sum_{n=0}^{\infty} A_n u_n(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta)$$

where the A_n's are found using the BC at $r = 1$ and orthogonality,

0.25 points $$g(\theta) = u(1,\theta) = \sum_{n=0}^{\infty} A_n \cos(n\theta)$$

Multiplying by $\cos(m\theta)$ for $m = 0, 1, 2, 3, \cdots$, and integrating from $\theta = 0$ to $\theta = \pi$ gives

$$\int_0^\pi g(\theta)d\theta = A_0\pi,$$
$$\int_0^\pi g(\theta)\cos(m\theta)d\theta = A_m\frac{\pi}{2}, \quad m = 1, 2, 3, \cdots .$$

Thus

0.5 points $$A_0 = \frac{1}{\pi}\int_0^\pi g(\theta)d\theta,$$

0.5 points $$A_m = \frac{2}{\pi}\int_0^\pi g(\theta)\cos(m\theta)d\theta, \quad m = 1, 2, 3, \cdots .$$

10 points 2. Solve the Heat Problem on the half unit disc

$$v_t = \Delta v, \quad 0 < r < 1, \quad 0 < \theta < \pi, \quad t > 0,$$

subject to inhomogeneous BCs

$$v(1,\theta,t) = g(\theta), \quad v(0,\theta,t) \text{ bounded }, \quad 0 < \theta < \pi, \quad t > 0,$$

$$\frac{\partial v}{\partial \theta}(r,0,t) = 0, \quad \frac{\partial v}{\partial \theta}(r,\pi,t) = 0, \quad 0 < r < 1, \quad t > 0,$$

and initial condition

$$v(r,\theta,0) = f(r,\theta), \quad 0 < r < 1, \quad 0 < \theta < \pi.$$

Your solution will have coefficients in terms of integrals involving $f(r,\theta)$.

Solutoin: First, let

0.5 points
$$V(r,\theta,t) = v(r,\theta,t) - u(r,\theta)$$

where $u(r,\theta)$ was found in Part 1. Then $V(r,\theta,t)$ satisfies the homogeneous problem,

0.25 points
$$V_t = \Delta V, \quad 0 < r < 1, \quad 0 < \theta < \pi, \quad t > 0,$$

subject to homogeneous BCs

0.25 points 0.25 points
$$V(1,\theta,t) = 0, \quad V(0,\theta,t) \text{ bounded }, \quad 0 < \theta < \pi, \quad t > 0,$$

0.25 points
$$\frac{\partial V}{\partial \theta}(r,0,t) = 0, \quad \frac{\partial V}{\partial \theta}(r,\pi,t) = 0, \quad 0 < r < 1, \quad t > 0,$$
0.25 points

and initial condition

0.25 points
$$V(r,\theta,0) = f(r,\theta) - u(r,\theta), \quad 0 < r < 1, \quad 0 < \theta < \pi.$$

We separate variables as

0.5 points
$$V(r,\theta,t) = R(r)H(\theta)T(t)$$

so that the PDE becomes

0.5 points
$$\frac{T'}{T} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2 H}\frac{d^2H}{d\theta^2} = -\lambda$$

where λ is constant since the l.h.s. depends only on t, and the r.h.s. on r, θ. Since the solution must decay, we expect $\lambda > 0$. Or we could argue this from general theory. From the middle equation, we have

0.5 points
$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r^2 = -\frac{1}{H}\frac{d^2H}{d\theta^2} = \mu$$

again, since the l.h.s. depends only on r and the r.h.s. only on θ.

Separating the BCs gives

0.25 points $V(1,\theta,t)=0 \quad \Rightarrow \quad R(1)=0$

0.25 points $V(0,\theta,t)$ bounded $\quad \Rightarrow \quad R(0)$ bounded

0.25 points $\dfrac{\partial V}{\partial \theta}(r,0,t)=0 \quad \Rightarrow \quad H'(0)=0$

0.25 points $\dfrac{\partial V}{\partial \theta}(r,\pi,t)=0 \quad \Rightarrow \quad H'(\pi)=0$

The problem for $H(\theta)$ is the same as before, thus

0.5 points $$H(\theta)=\cos(n\theta), \quad \mu=n^2, \quad n=0,1,2,3,\cdots.$$

Thus, the equation for $R(r)$ is

0.5 points $$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\left(\lambda r^2-n^2\right)R=0,$$

and hence the general solution is

$$R(r)=c_1 J_n\left(\sqrt{\lambda}r\right)+c_2 Y_n\left(\sqrt{\lambda}r\right)$$

0.25 points Since $R(0)$ must be bounded, $c_2=0$. The other BC is

0.25 points $$0=R(1)=c_1 J_n\left(\sqrt{\lambda}\right)$$

0.5 points Thus $\lambda_{nm}=j_{n,m}^2$, where $j_{n,m}$ is the m'th zero of J_n.

Thus, for $n=0,1,2,3,\cdots$ and $m=1,2,3,\cdots$,

0.5 points $$V_{nm}(r,\theta,t)=J_n(rj_{n,m})\cos(n\theta)e^{-\lambda_{nm}t}$$

solves the PDE and BCs. To solve the IC, we sum over n, m and use superposition,

1 point $$V(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_{nm}V_{nm}(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_{nm}J_n(rj_{n,m})\cos(n\theta)e^{-\lambda_{nm}t}$$

where the A_{nm}'s are found from the IC and orthogonality:

0.5 points $$f(r,\theta)-u(r,\theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}A_{nm}J_n(rj_{n,m})\cos(n\theta)$$

0.5 points Multiplying by $rJ_k(rj_{k,l})\cos(k\theta)$, for $k = 0, 1, 2, 3, \cdots$ and $l = 1, 2, 3, \cdots$, and integrating in r, θ, we have

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi}\big(f(r,\theta)-u(r,\theta)\big)rJ_0(rj_{0,l})drd\theta = A_{0l}\pi\int_{r=0}^{1} r\big(J_0(rj_{0,l})\big)^2 dr$$

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi}\big(f(r,\theta)-u(r,\theta)\big)rJ_k(rj_{k,l})\cos(k\theta)drd\theta = A_{kl}\frac{\pi}{2}\int_{r=0}^{1} r\big(J_k(rj_{k,l})\big)^2 dr, \quad k \geq 1$$

Thus

0.5 points
$$A_{0l} = \frac{1}{\pi}\frac{\int_{r=0}^{1}\int_{\theta=0}^{\pi}\big(f(r,\theta)-u(r,\theta)\big)rJ_0(rj_{0,l})drd\theta}{\int_{r=0}^{1} r\big(J_0(rj_{0,l})\big)^2 dr}, \quad l \geq 1$$

0.5 points
$$A_{kl} = \frac{2}{\pi}\frac{\int_{r=0}^{1}\int_{\theta=0}^{\pi}\big(f(r,\theta)-u(r,\theta)\big)rJ_k(rj_{k,l})\cos(k\theta)drd\theta}{\int_{r=0}^{1} r\big(J_k(rj_{k,l})\big)^2 dr}, \quad k \geq 1, l \geq 1.$$

Finally,

$$v(r,\theta,t) = V(r,\theta,t) + u(r,\theta)$$

4 points

3. Prove the solution to 2 is unique.
   Hint: You'll need to use a result derived from the Divergence Theorem (on the given page). You don't need to consider r, θ: denoting the region by D and using dV will work fine.

   0.25 points

   Solutoin: Take two solutions v_1 and v_2. Define the difference $w = v_1 - v_2$. Note that w satisfies

   0.25 points $$w_t = \Delta w, \quad 0 < r < 1, \quad 0 < \theta < \pi, \quad t > 0,$$

   subject to the homogeneous BCs

   $$w(1, \theta, t) = 0, \quad w(0, \theta, t) \text{ bounded }, \quad 0 < \theta < \pi, \quad t > 0,$$

   0.25 points $$\frac{\partial w}{\partial \theta}(r, 0, t) = 0, \quad \frac{\partial w}{\partial \theta}(r, \pi, t) = 0, \quad 0 < r < 1, \quad t > 0,$$

   and the homogeneous initial condition

   0.25 points $$w(r, \theta, 0) = 0, \quad 0 < r < 1, \quad 0 < \theta < \pi.$$

   Define the mean square difference between solutions,

   0.5 points $$E(t) = \iint_D w(t)^2 dV \geq 0$$

   Differentiate in time,

   0.5 points $$\frac{dE(t)}{dt} = \iint_D 2ww_t dV = 2\iint_D w\Delta w dV$$

   0.5 points $$= -2\iint_D |\nabla w|^2 dV + 2\int_{\partial D} w\nabla w \cdot n dS$$

   0.5 points But $w = 0$ on the boundary where $r = 1$ and $\nabla w \cdot n = 0$ on the boundary where $\theta = 0, \pi$, so that

   $$\frac{dE(t)}{dt} = -2\iint_D |\nabla w|^2 dV \leq 0$$

   0.5 points Note that at $t = 0$

   $$E(0) = \iint_D w(0)^2 dV = 0$$

   0.5 points Thus, $E(t)$ is a non-negative, non-increasing function , that starts at zero. Hence $E(t) = 0$ for all time, which implies by continuity that $w(r, \theta, t) = 0$ for all r, θ, t. Hence $v_1 = v_2$, and the solution to 2 is unique.