Section 4.1 Problem Notes

4.1.4. This was using the geometric understanding of the determinant of $A = \begin{pmatrix} u \\ v \end{pmatrix} \in M_2(\mathbb{F})$ as the signed area of the parallelogram generated by two vectors in $\mathbb{F}^2$: $\det A = O(\begin{pmatrix} u \\ v \end{pmatrix})\text{Area}(u, v)$ with $O(\begin{pmatrix} u \\ v \end{pmatrix}) = \pm 1$ depending on how u, v are oriented. Since area is physical all of these numbers should be non-negative.

Section 4.2

4.2.23. This is the type of problem that is nicely done by induction. Some of your answers were sort of lazy about this method so look at the last problem for a demonstration of how to do this type of proof.

4.3

4.3.4. Be sure that you are confident with doing determinant calculations of 3×3 matrices. Some of you left of the important alternating sign term in the cofactor expansion, e.g., the $(-1)^{1+j}$ term when you calculate the determinant of $A = (A_{ij})$ via cofactor expansion along the first row:

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j}).$$

4.3.13.

(a) We want to show that $\det(\overline{M}) = \overline{\det(M)}$ for $M \in M_n(\mathbb{C})$.

The important fact about complex numbers here is that when $z, w \in \mathbb{C}$, we have $\overline{z} \cdot \overline{w} = \overline{z \cdot w}$ and $\overline{z + w} = \overline{z} + \overline{w}$.

Proof. We prove this by induction.

For the base case: $k = 1$. Then $M = (z)$ is just a complex number and trivially $\det(\bar{(M)}) = \det(\overline{z}) = \overline{z} = \overline{\det(M)}$.

For induction assume that when $1 \leq k \leq n - 1$ than $A \in M_k(\mathbb{C})$ has $\det(\overline{A}) = \overline{\det(A)}$.

Now we show that the claim holds for $k = n$. By cofactor expansion about the first row:

$$
\begin{aligned}
\det(\overline{M}) &= \sum_{j=1}^{n}(-1)^{1+j}\overline{M_{1j}}\det(\tilde{\overline{M}}_{1j}) && \text{, by def. of determinant and } \overline{M} \\
&= \sum_{j=1}^{n}(-1)^{1+j}\overline{M_{1j}}\det(\overline{\tilde{M}_{1j}}) && \text{, since taking } \tilde{A}_{ij} \text{ and taking the conjugate commute} \\
&= \sum_{j=1}^{n}(-1)^{1+j}\overline{M_{1j}} \cdot \overline{\det(\tilde{M}_{1j})} && \text{, by induction since } \tilde{M}_{1j} \in M_{n-1}(\mathbb{C}). \\
&= \sum_{j=1}^{n}(-1)^{1+j}\overline{M_{1j} \cdot \det(\tilde{M}_{1j})} && \text{, because } \tilde{M}_{1j}, M_{1j} \in \mathbb{C} \\
&= \sum_{j=1}^{n}\overline{(-1)^{1+j}M_{1j} \cdot \det(\tilde{M}_{1j})} && \text{, since } (-1)^{1+j}, M_{1j}\det(\tilde{M}_{1j}) \in \mathbb{C} \text{ and } (-1)^{1+j} = \overline{(-1)^{1+j}} \\
&= \overline{\sum_{j=1}^{n}(-1)^{1+j}M_{1j} \cdot \det(\tilde{M}_{1j})} && \text{, because each term of the summand is a complex number} \\
&= \overline{\det(M)}
\end{aligned}
$$

$\square$

(b) This part comes from three facts:

- $\det A^t = \det A$
- $\det(\overline{A}) = \overline{\det(A)}$ by part (a).
- For $z \in \mathbb{C}$, $z\overline{z} = |z|^2$ and $|z| \in \mathbb{R}, |z| \geq 0$.

If Q is unitary, then $QQ^* = I$.

Hence $1 = \det I = \det QQ^* = \det Q \det Q^* = \det Q \det \overline{Q^t} = \det Q \overline{\det Q^t} = \det Q \overline{\det Q} = |\det Q|^2$.

Thus $|\det Q|^2 = 1$ and since $|\det Q| \in \mathbb{R}$[1] , $|\det Q| \geq 0$ we have $|\det Q| = 1$.

[1]In contrast $\det Q$ may be in $\mathbb{C} \setminus \mathbb{R}$