Math 4377/6308 Advanced Linear Algebra

Chapter 2 Review and Solution to Problems

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Pb 2.1.15

Recall the definition of $P(\mathbb{R})$ on page 10. Define

$$T : P(\mathbb{R}) \to P(\mathbb{R}) \quad \text{by} \quad T(f(x)) = \int_0^x f(t)dt.$$

Prove that T is linear and one-to-one, but not onto.

T is linear, since with $f, g \in \mathrm{P}(R)$ and $c \in R$:

$$T(cf+g) = \int_0^x (cf(t)+g(t))\,dt = c\int_0^x f(t)\,dt + \int_0^x g(t)\,dt = cT(f)+T(g).$$

T is one-to-one, since if $\mathrm{T}(f) = 0$, differentiation of both sides gives $f = 0$ so the null-space is trivial. T is not onto, since all polynomials $\mathrm{T}(f)$ are zero at $x = 0$, and can therefore not be equal to any polynomial with a nonzero constant term.

Pb 2.1.35(a)

Let V be a finite-dimensional vector space and $T : V \to V$ be linear.

(a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.

Be careful to say in each part where finite-dimensionality is used.

(a) Suppose V=R(T)+N(T). Let β, γ be bases for R(T),N(T), so $\mathrm{V} = \mathrm{span}(\{\beta \cup \gamma\})$. By the dimension theorem (using finite-dimensionality), $\dim(\mathrm{N(T)}) + \dim(\mathrm{R(T)}) = \dim(\mathrm{V})$, so there are exactly $\dim(\mathrm{V})$ vectors in $\beta \cup \gamma$. By Corollary 2 to the Replacement Theorem, $\beta \cup \gamma$ is then a basis for V, which means that the vectors are linearly independent and $\mathrm{R(T)} \cap \mathrm{N(T)} = \{\mathit{0}\}$, that is, V=R(T) $\oplus$ N(T).

Pb 2.1.35(a)

(b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.

(b) Suppose $R(T) \cap N(T) = \{0\}$. Let β, γ be bases for R(T),N(T). The set $\beta \cup \gamma$ is then linearly independent (since only the zero vector can be written as linear combinations of vectors in β or in γ). By the dimension theorem (using finite-dimensionality), $\dim(N(T)) + \dim(R(T)) = \dim(V)$, so there are exactly $\dim(V)$ vectors in $\beta \cup \gamma$. By Corollary 2 to the Replacement Theorem, $\beta \cup \gamma$ is then a basis for V, so $V = R(T) \oplus N(T)$.

Pb 2.2.12

Let V be a finite-dimensional vector space and T be the projection on W along W' where W and W' are subspaces of V such that $V = W \oplus W'$. (Recall that a function $T : V \to V$ is called the projection on W along W' if, for $x = x_1 + x_2$ with $x_1 \in W$ and $x_2 \in W'$, we have $T(x) = x_1$. Find an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

From the definition of projection, $\mathsf{V}=\mathsf{W}\oplus\mathsf{W}'$ and for $x = x_1 + x_2$ with $x_1 \in \mathsf{W}_1$, $x_2 \in \mathsf{W}_2$, we have $\mathsf{T}(x) = x_1$. Let $\beta_{\mathsf{W}} = \{u_1, \ldots, u_k\}$ be a basis for W and $\beta = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ be its extension to a basis for V. Then $\{u_{k+1}, \ldots, u_n\}$ is a basis for W′ (it spans because $\mathsf{V}=\mathsf{W}\oplus\mathsf{W}'$, and it is linearly independent since it is a subset of another basis). Now, we have $T(u_i) = u_i$ for $i = 1, \ldots, k$ and $T(u_i) = \mathit{0}$ for $i = k+1, \ldots, n$, so

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Pb 2.3.17

Let V be a vector space. Determine all linear transformations $T : V \to V$ such that $T = T^2$. Hint: Note that $x = T(x) + (x - T(x))$ for every $x \in V$, and show that $V = \{y : T(y) = y\} \oplus N(T)$ (see the exercises of Section 1.3).

For any $x \in \mathsf{V}$, we have $x = \mathsf{T}(x) + (x - \mathsf{T}(x))$. Here, $\mathsf{T}(x) \in \mathsf{R}(\mathsf{T})$ and $x - \mathsf{T}(x) \in \mathsf{N}(\mathsf{T})$, since

$$\mathsf{T}(x - \mathsf{T}(x)) = \mathsf{T}(x) - \mathsf{T}^2(x) = \mathsf{T}(x) - \mathsf{T}(x) = \mathit{0}.$$

This gives V=R(T)+N(T), and by exercise 2.1.35 (a), V=R(T) $\oplus$ N(T). By exercise 2.1.26, then $\mathsf{V} = \{y : \mathsf{T}(y) = y\} \oplus \mathsf{N}(\mathsf{T})$.

This characterizes all maps T such that $\mathsf{T}^2 = \mathsf{T}$, since for any direct sum $\mathsf{V} = \mathsf{V}_1 \oplus \mathsf{V}_2$, the transformation $\mathsf{T}(v) = v_1$ where $v = v_1 + v_2, v_1 \in \mathsf{V}_1, v_2 \in \mathsf{V}_2$ satisfies $\mathsf{T}^2 = \mathsf{T}$, acts as the identity on its range V_1, and has the null space V_2.

Pb 2.4.17(a)

Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(a) $\mathsf{T}(\mathsf{V}_0)$ contains 0_W, since $0_\mathsf{V} \in \mathsf{V}_0$ and $\mathsf{T}(0_\mathsf{V}) = 0_\mathsf{W}$. Let $u_1, u_2 \in \mathsf{T}(\mathsf{V}_0)$, then there exist $v_1, v_2 \in \mathsf{V}_0$ such that $\mathsf{T}(v_1) = u_1$ and $\mathsf{T}(v_2) = u_2$. But then $v_1 + v_2 \in \mathsf{V}_0$ and $\mathsf{T}(v_1 + v_2) = \mathsf{T}(v_1) + \mathsf{T}(v_2) = u_1 + u_2 \in \mathsf{T}(\mathsf{V}_0)$. Similarly for scalar multiplication, showing that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.

Pb 2.4.17(b)

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

(b) Let $\beta = \{u_1, \ldots, u_n\}$ be a basis for V_0. $\mathsf{T}(\beta)$ is then a basis for $\mathsf{T}(\mathsf{V}_0)$, since it spans $\mathsf{T}(\mathsf{V}_0)$ and its vectors are linearly independent:

$$a_1\mathsf{T}(u_1) + \cdots + a_n\mathsf{T}(u_n) = \mathsf{T}(a_1u_1 + \cdots + a_nu_n) = 0$$

gives $a_1u_1 + \cdots + a_nu_n = 0$ since T is an isomorphism, and $a_1 = \cdots = a_n = 0$ since β is a basis for V_0. Thus, $n = \dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

Pb 2.5.8

Prove the following generalization of Theorem 2.33. Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Then $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma}Q$, where Q is the matrix that changes β'-coordinates into β-coordinates and P is the matrix that changes γ'-coordinates into γ-coordinates.

We have $Q = [\mathsf{I_V}]_{\beta'}^{\beta}$ and $P = [\mathsf{I_W}]_{\gamma'}^{\gamma}$. Then $\mathsf{T} = \mathsf{I_W T} = \mathsf{T I_V}$ and

$$P[\mathsf{T}]_{\beta'}^{\gamma'} = [\mathsf{I_W}]_{\gamma'}^{\gamma}[\mathsf{T}]_{\beta'}^{\gamma'} = [\mathsf{I_W T}]_{\beta'}^{\gamma} = [\mathsf{T I_V}]_{\beta'}^{\gamma} = [\mathsf{T}]_{\beta}^{\gamma}[\mathsf{I_V}]_{\beta'}^{\beta} = [\mathsf{T}]_{\beta}^{\gamma}Q.$$

Therefore, $[\mathsf{T}]_{\beta'}^{\gamma'} = P^{-1}[\mathsf{T}]_{\beta}^{\gamma}Q$.