Math 4377/6308 Advanced Linear Algebra

Chapter 3 Review and Solution to Problems

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math4377

Pb 3.2.14

Let $T, U : V \to W$ be linear transformations.

(a) Prove that $R(T+U) \subseteq R(T) + R(U)$.

(b) Prove that if W is finite-dimensional, then $\mathrm{rank}(T+U) \leq \mathrm{rank}(T) + \mathrm{rank}(U)$.

(c) Deduce from (b) that $\mathrm{rank}(A+B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$ for any $m \times n$ matrices A and B.

(a) $\forall w \in R(T+U)$, $\exists v \in V$ such that $w = (T+U)(v) \overset{?}{=} T(v) + U(v) \in R(T) + R(U)$.

(b) $\mathrm{rank}(T+U) = \dim(R(T+U)) \overset{?}{\leq} \dim(R(T)+R(U)) \overset{?}{=} \dim(R(T)) + \dim(R(U)) - \dim(R(T) \cap R(U)) \leq \dim(R(T)) + \dim(R(U)) = \mathrm{rank}(T) + \mathrm{rank}(U)$.

(c) Note that $L_{A+B} \overset{?}{=} L_A + L_B$. Then $\mathrm{rank}(A+B) = \mathrm{rank}(L_{A+B}) = \mathrm{rank}(L_A + L_B) \leq \mathrm{rank}(L_A) + \mathrm{rank}(L_B) = \mathrm{rank}(A) + \mathrm{rank}(B)$.

Pb 3.2.17

Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exists a 3×1 matrix B and a 1×3 matrix C such that $A = BC$.

$(\Rightarrow)$ If $B = b \in F^{3\times1}$ and $C = c^T = (c_1, c_2, c_3) \in F^{1\times3}$), then $BC = bc^T = b(c_1, c_2, c_3) \stackrel{?}{=} (c_1 b, c_2 b, c_3 b)$, thus $\mathrm{range}(BC) \stackrel{?}{=} \mathrm{span}(\{b\})$ and $\mathrm{rank}(BC) = \dim(\mathrm{span}(\{b\})) \stackrel{?}{\leq} 1$.

$(\Leftarrow)$ If $A = (a_1, a_2, a_3) \in F^{3\times3}$ have rank 1, then $\mathrm{range}(A) \stackrel{?}{=} \mathrm{span}(\{a_1, a_2, a_3\}) \stackrel{?}{=} \mathrm{span}(\{b\})$ for some $b \neq 0 \in F^{3\times1}$, and $\exists c^T = (c_1, c_2, c_3) \in F^{1\times3}$ such that $a_1 = c_1 b$, $a_2 = c_2 b$, $a_3 = c_3 b$. Therefore, $A = (c_1 b, c_2 b, c_3 b) = b(c_1, c_2, c_3) = BC$ with $B = b$ and $C = c^T$.

Pb 3.2.19

Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.

If A is $m \times n$ with $\mathrm{rank}(A) = m$ and B is $n \times p$ with $\mathrm{rank}(B) = n$, then $L_A : F^n \to F^m$ with $R(A) = R(L_A) = L_A(F^n) \stackrel{?}{=} F^m$ and $L_B : F^p \to F^n$ with $R(B) = R(L_B) = L_B(F^p) \stackrel{?}{=} F^n$.

Note that AB is $m \times p$ and $L_{AB} : F^p \to F^m$. Therefore, $R(AB) = R(L_{AB}) = L_{AB}(F^p) \stackrel{?}{=} L_A(L_B(F^p)) \stackrel{?}{=} L_A(F^n) \stackrel{?}{=} F^m$ and $\mathrm{rank}(AB) \stackrel{?}{=} m$.