Pb 3.2.14

Let \( T, U : V \to W \) be linear transformations.

(a) Prove that \( R(T + U) \subseteq R(T) + R(U) \).

(b) Prove that if \( W \) is finite-dimensional, then 
    \[ \text{rank}(T + U) \leq \text{rank}(T) + \text{rank}(U). \]

(c) Deduce from (b) that \( \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \) for
    any \( m \times n \) matrices \( A \) and \( B \).

(a) \( \forall w \in R(T + U), \exists v \in V \) such that
    \[ w = (T + U)(v) = T(v) + U(v) \in R(T) + R(U). \]

(b) \( \text{rank}(T + U) = \dim(R(T + U)) \leq \dim(R(T) + R(U)) \leq \dim(R(T)) + \dim(R(U)) - \dim(R(T) \cap R(U)) \leq \dim(R(T)) + \dim(R(U)) = \text{rank}(T) + \text{rank}(U). \)

(c) Note that \( L_{A+B} = L_A + L_B \). Then 
    \[ \text{rank}(A + B) = \text{rank}(L_{A+B}) = \text{rank}(L_A + L_B) \leq \text{rank}(L_A) + \text{rank}(L_B) = \text{rank}(A) + \text{rank}(B). \]
Pb 3.2.17

Prove that if $B$ is a $3 \times 1$ matrix and $C$ is a $1 \times 3$ matrix, then the $3 \times 3$ matrix $BC$ has rank at most 1. Conversely, show that if $A$ is any $3 \times 3$ matrix having rank 1, then there exists a $3 \times 1$ matrix $B$ and a $1 \times 3$ matrix $C$ such that $A = BC$.

($\Rightarrow$) If $B = b \in F^{3 \times 1}$ and $C = c^T = (c_1, c_2, c_3) \in F^{1 \times 3}$, then $BC = bc^T = b(c_1, c_2, c_3) = (c_1b, c_2b, c_3b)$, thus $\text{range}(BC) = \text{span}\{b\}$ and $\text{rank}(BC) = \dim(\text{span}\{b\}) \leq 1$.

($\Leftarrow$) If $A = (a_1, a_2, a_3) \in F^{3 \times 3}$ have rank 1, then $\text{range}(A) = \text{span}\{a_1, a_2, a_3\} = \text{span}\{b\}$ for some $b \neq 0 \in F^{3 \times 1}$, and $\exists c^T = (c_1, c_2, c_3) \in F^{1 \times 3}$ such that $a_1 = c_1b$, $a_2 = c_2b$, $a_3 = c_3b$. Therefore, $A = (c_1b, c_2b, c_3b) = b(c_1, c_2, c_3) = BC$ with $B = b$ and $C = c^T$. 
Pb 3.2.19

Let $A$ be an $m \times n$ matrix with rank $m$ and $B$ be an $n \times p$ matrix with rank $n$. Determine the rank of $AB$. Justify your answer.

If $A$ is $m \times n$ with $\text{rank}(A) = m$ and $B$ is $n \times p$ with $\text{rank}(B) = n$, then $L_A : F^n \to F^m$ with $R(A) = R(L_A) = L_A(F^n) = F^m$ and $L_B : F^p \to F^n$ with $R(B) = R(L_B) = L_B(F^p) = F^n$.

Note that $AB$ is $m \times p$ and $L_{AB} : F^p \to F^m$. Therefore, $R(AB) = R(L_{AB}) = L_{AB}(F^p) = L_A(L_B(F^p)) = L_A(F^n) = F^m$ and $\text{rank}(AB) = m$. 