Math 4377/6308 Advanced Linear Algebra
Chapter 4 Review and Solution to Problems

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/~jiwenhe/math4377
Let $A \in M_n(F)$ be an upper triangular matrix. Show that

$$\det(A) = \prod_{i=1}^{n} A_{ii}.$$ 

First, note that $A_{ij} = 0$ for all $i > j$. We proceed by induction on $n$.

- $n = 1$: Obvious as $\det(A) = A_{11}$.
- $n - 1 \Rightarrow n$: We take the determinant by expanding along the last row of $A$. Let $\tilde{A}_{ij}$ be the matrix obtained from $A$ by deleting the $i$th row and $j$th column. Then

$$\det(A) = (-1)^{n+n} A_{nn} \det(\tilde{A}_{nn}) = A_{nn} \prod_{i=1}^{n-1} A_{ii} = \prod_{i=1}^{n} A_{ii}$$

by the induction hypothesis as $\tilde{A}_{nn}$ is upper triangular.
Suppose $M \in M_n(F)$ is nilpotent, i.e., there is a $k \geq 0$ such that $M^k = 0$. Show $M$ is not invertible.

Proof. By the fact that $\det(AB) = \det(A) \det(B)$ and an easy induction argument, we have

$$0 = \det(0) = \det(M^k) = \prod_{i=1}^{k} \det(M) = (\det(M))^k.$$ 

Taking $k$th roots, we have $\det(M) = 0$, so $M$ is not invertible.
Suppose $M \in M_n(F)$ is skew-symmetric, i.e., $M^t = -M$. Show that if $n$ is odd, then $M$ is not invertible. What if $n$ is even?

Proof. We know that

$$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M).$$

If $n$ is odd, then $\det(M) = -\det(M)$, so $\det(M) = 0$, and $M$ is not invertible. However, this equation tells us nothing if $n$ is even. To fully answer this question, we must include examples of skew symmetric matrices which are invertible and non-invertible for $n$ even. For $n = 2$, both

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are skew symmetric; the first is not invertible, and the second is invertible. Now for $n > 2$, we get examples by taking block diagonal matrices using the above examples.
Pb 4.3.15

Prove that if \( A, B \in M_n(F) \) are similar, then \( \det(A) = \det(B) \).

Proof. If \( A \) and \( B \) are similar, then

\[
A = Q^{-1}BQ
\]

for some invertible \( Q \). Taking determinants gives

\[
\det(A) = \det(Q^{-1}BQ) = \det(Q^{-1}) \det(B) \det(Q)
\]

\[
= \frac{1}{\det(Q)} \det(B) \det(Q) = \det(B).
\]
Pb 4.3.21

Suppose that \( M \in M_n(F) \) can be written in the block upper triangular form

\[
M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}
\]

where \( A \in M_k(F) \) and \( C \in M_{n-k}(F) \). Prove that

\[
\det(M) = \det(A) \det(C).
\]

First, we proceed by induction on \( n \).

- \( n = 2 \): Obvious as

\[
\det(M) = \det \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac.
\]

- \( n - 1 \Rightarrow n \): We take the determinant by expanding along the first column of \( M \). Let \( \hat{M}_{ij} \) be the matrix obtained from \( M \) by deleting the \( i \)th row and \( j \)th column.
First, note that $M_{i1} = 0$ for all $i > k$. For $i \leq k$, $M_{i1} = A_{i1}$ and
\[
\det \tilde{M}_{i1} = \det \begin{pmatrix} \tilde{A}_{i1} & \tilde{B} \\ 0 & C \end{pmatrix} = \det(\tilde{A}_{i1}) \det(C)
\]
by the induction hypothesis as $\tilde{M}_{i1}$ is block upper triangular. Then
\[
\det(M) = \sum_{i=1}^{n} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1}) = \sum_{i=1}^{k} (-1)^{i+1} M_{i1} \det(\tilde{M}_{i1})
\]
\[
= \left( \sum_{i=1}^{k} (-1)^{i+1} A_{i1} \det(\tilde{A}_{i1}) \right) \det(C) = \det(A) \det(C).
\]