Math 4377/6308 Advanced Linear Algebra
Chapter 5  Review and Solution to Problems

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Prove that the eigenvalues of an upper triangular matrix $A$ are the diagonal entries of $A$.

Let $A$ be an upper triangular matrix. Notice that $\lambda I_n$ is also an upper triangular matrix, thus $A - \lambda I_n$ is upper triangular. From problem 4.2.23 (which we proved on a previous homework) we know that the $\det(A - \lambda I_n)$ is the product of the diagonal entries, giving

$$p(\lambda) = \det(A - \lambda I_n) = \prod_{i=1}^{n} (a_{ii} - \lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

where $a_{ii}$ are the diagonal entries of $A$. This is the characteristic polynomial of $A$ and its roots are $a_{ii}$ for all $i$. Thus the eigenvalues of $A$ are its diagonal entries.
Pb 5.1.12

(a) Prove that similar matrices have the same characteristic polynomial.

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space $V$ is independent of the choice of basis for $V$.

(a) Let $A$ and $B$ be similar, i.e., $\exists Q$ invertible such that $B = Q^{-1}AQ$. Note that $\det(Q^{-1}) = (\det(Q))^{-1}$. We have

\[
p_B(\lambda) = \det(B - \lambda I_n) = \det(Q^{-1}AQ - \lambda I_n) = \det(Q^{-1}AQ - \lambda Q^{-1}I_nQ) \\
= \det(Q^{-1}(A - \lambda I_n)Q) = \det(Q^{-1}) \det(A - \lambda I_n) \det(Q) \\
= \det(A - \lambda I_n)(\det(Q))^{-1} \det(Q) = \det(A - \lambda I_n) = p_A(\lambda).
\]

(b) Let $T$ be a linear operator on a finite-dimensional vector space $V$, and let $\beta$ and $\gamma$ are any ordered bases for $V$, then $[T]_{\beta}$ is similar to $[T]_{\gamma}$. Result follows by (a).
Pb 5.1.14

For any square matrix $A$, prove that $A$ and $A^t$ have the same characteristic polynomial (and hence the same eigenvalues).

We know that $\det(A^t) = \det(A)$ so a simple calculation gives

$$p_A(\lambda) = \det(A - \lambda I_n) = \det((A - \lambda I_n)^t) = \det(A^t - \lambda I_n) = p_{A^t}(\lambda),$$

since $\lambda I_n$ is symmetric. Thus $A$ and $A^t$ have the same characteristic polynomial.
Pb 5.1.20

Let $A$ be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$ 

Prove that $f(0) = a_0 = \det(A)$. Deduce that $A$ is invertible if and only if $a_0 \neq 0$.

Note that

$$f(t) = \det(A - tl_n) \implies f(0) = \det(A - 0 \cdot l_n) = \det(A).$$

Also, we have

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \implies f(0) = a_0.$$

Thus $a_0 = \det(A)$. From Corollary of Theorem 4.7, $A$ is invertible if and only if $\det(A) = a_0 \neq 0$. 
Pb 5.2.7

For \( A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \), find an expression for \( A^n \), where \( n \) is an arbitrary positive integer.

Note that \( A \) has two distinct eigenvalues 5 and \(-1\), thus is diagonalizable, i.e.,

\[
Q^{-1}AQ = D \quad \text{with} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.
\]

Note that \( Q^{-1} = \frac{1}{3}Q \). So we have

\[
A^n = QD^nQ^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 5^n + 2 \cdot (-1)^n & 2 \cdot 5^n + 2 \cdot (-1)^{n+1} \\ 5^n + (-1)^{n+1} & 2 \cdot 5^n + (-1)^n \end{pmatrix}.
\]
Pb 5.2.12

Let \( T \) be an invertible linear operator on a finite-dimensional vector space \( \mathcal{V} \).

(a) Recall that for any eigenvalue \( \lambda \) of \( T \), \( \lambda^{-1} \) is an eigenvalue of \( T^{-1} \). Prove that the eigenspace of \( T \) corresponding to \( \lambda \) is the same as the eigenspace of \( T^{-1} \) corresponding to \( \lambda^{-1} \).

(b) Prove that if \( T \) is diagonalizable, then \( T^{-1} \) is diagonalizable.

(a) Let \( E_{T,\lambda} = N(T - \lambda I_V) \) and \( E_{T^{-1},\lambda^{-1}} = N(T^{-1} - \lambda^{-1} I_V) \).

\((E_{T,\lambda} \subseteq E_{T^{-1},\lambda^{-1}})\) If \( x \in E_{T,\lambda} \), then \( T(x) = \lambda x \). Applying \( T^{-1} \) to both sides gives

\[ x = T^{-1}(T(x)) = T^{-1}(\lambda x) = \lambda T^{-1}(x) \implies T^{-1}(x) = \lambda^{-1} x, \]

since \( T \) is invertible, \( \lambda \neq 0 \).

\((E_{T^{-1},\lambda^{-1}} \subseteq E_{T,\lambda})\) If \( x \in E_{T^{-1},\lambda^{-1}} \), then \( T^{-1}(x) = \lambda^{-1} x \). Applying \( T \) to both sides gives

\[ x = T(T^{-1}(x)) = T(\lambda^{-1} x) = \lambda^{-1} T(x) \implies T(x) = \lambda x. \]
(b) If $T$ is diagonalizable, then there is a basis $\beta$ for $V$ such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Since $T$ is invertible, $\lambda_j \neq 0$. From Theorem 2.18, we have

$$[T^{-1}]_\beta = ([T]_\beta)^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 \\ \vdots & \ddots \\ 0 & \cdots & \lambda_n^{-1} \end{pmatrix}$$

Since $[T^{-1}]_\beta$ is diagonal, $T^{-1}$ is diagonalizable.
Let \( A \in M_{n \times n}(F) \). Recall from Exercise 14 of Section 5.1 that \( A \) and \( A^t \) have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue \( \lambda \) of \( A \) and \( A^t \), let \( E_\lambda \) and \( E'_\lambda \) denote the corresponding eigenspaces for \( A \) and \( A^t \), respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue \( \lambda \), \( \dim(E_\lambda) = \dim(E'_\lambda) \).

(c) Prove that if \( A \) is diagonalizable, then \( A^t \) is also diagonalizable.

(a) Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \). The eigenvalues of \( A \) are \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \), and we have \( E_{\lambda_1} = \text{span}\{(1, 0)^t\} \) and \( E_{\lambda_2} = \text{span}\{(1, 1)^t\} \). However, we have \( A^t = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \) and \( E_{\lambda_1} = \text{span}\{(1, -1)^t\} \) and \( E_{\lambda_2} = \text{span}\{(0, 1)^t\} \).
(b) Note that $E_\lambda = N(A - \lambda I)$ and $E'_\lambda = N(A^t - \lambda I)$. By the dimension theorem (Theorem 2.3), we have

$$\dim(E_\lambda) = \dim(N(A - \lambda I)) = n - \text{rank}(A - \lambda I) = n - \text{rank}((A - \lambda I)^t)$$

$$= n - \text{rank}(A^t - \lambda I) = \dim(N(A^t - \lambda I)) = \dim(E'_\lambda)$$

since $\text{rank}B^t = \text{rank}B$ for any matrix $B$. 
(c) If $A$ is diagonalizable, then from Theorem 5.6, the characteristic polynomial of $A$ splits. Let $m_\lambda$ be the multiplicity of $\lambda$ as an eigenvalue of $A$. From Theorem 5.9, we have $\dim(E_\lambda) = m_\lambda$. Note that $m_\lambda$ is also the multiplicity of $\lambda$ as an eigenvalue of $A'$. From (a), we have $\dim(E_\lambda') = \dim(E_\lambda) = m_\lambda$. From Theorem 5.9, $A^t$ is diagonalizable.
Pb 5.2.18

(a) Prove that if $T$ and $U$ are simultaneously diagonalizable operators, then $T$ and $U$ commute (i.e., $UT = TU$).

(b) Prove that if $A$ and $B$ are simultaneously diagonalizable matrices, then $A$ and $B$ commute (i.e., $AB = BA$).

(a) Note that if $D_1$ and $D_2$ are diagonal matrices, then $D_1 D_2 = D_2 D_1$. If $T$ and $U$ are simultaneously diagonalizable operators, then there is a basis $\beta$ for $V$ such that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Using that fact and Theorem 2.11, we get


By Theorem 2.20, we can conclude from $[TU]_\beta = [UT]_\beta$ that $TU = UT$. 
(b) If $A$ and $B$ are simultaneously diagonalizable matrices, then there is an invertible matrix such that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal. As noted above, this means that these matrices commute. Then

$$Q^{-1}ABQ = (Q^{-1}AQ)(Q^{-1}BQ) = (Q^{-1}BQ)(Q^{-1}AQ) = Q^{-1}BAQ.$$ 

Multiplying the above by $Q$ on the left and $Q^{-1}$ on the right gives $AB = BA$. 