Advanced Linear Algebra

Midterm 1 **Math 4377 / 6308 (Spring 2015)** **March 5, 2015**

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20 points 1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case).

(1) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

(2) Any set containing the zero vector is linearly dependent.

(3) Subsets of linearly dependent sets are linearly dependent.

(4) Subsets of linearly independent sets are linearly independent.

(5) Every vector space that is generated by a finite set has a basis.

(6) Every vector space has a finite basis.

(7) If a vector space has a finite basis, then the number of vectors in every basis is the same.

(7') The dimension of $M_{m\times n}(F)$ is $m+n$.

(8) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.

(9) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

(10) If $T : V \to W$ is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$.

20 points 2. The first four Chebyshev polynomials are 1, x, $2x^2-1$, and $4x^3-3x$. These polynomials arise naturally in the study of certain important differential equations. Show that the first four Chebyshev polynomials form a basis of $P_3(\mathbb{R})$.

20 points 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(a, b, c) = (3a, -2a + c, b).$$

Prove that T is an isomorphism and find T^{-1}.

20 points 4. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(a, b, c) = (a - b, 2c)$.

(a) Show that T is a linear transformation.

(b) Find bases for the null space and the range of T.

(c) Compute the nullity and rank of T, and verify the dimension theorem.

20 points 5. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(c) Suppose

$$W_1 = \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}, \quad W_2 = \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_q\}.$$

where $\mathbf{u}_1, \cdots, \mathbf{u}_p$ and $\mathbf{v}_1, \cdots, \mathbf{v}_q$ are vectors in V. Show that

$$W_1 + W_2 = \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}.$$

20 points

6. (**BONUS PROBLEM**) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

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Problem 1.

(1) False.

(2) True.

(3) , False.

(4) True.

(5) True.

(6) False.

(7) True.

(7') False.

(8) True.

(9) True.

(10) False.

Problem 2.

Let $\beta = \{1, x, 2x^2 - 1, 4x^3 - 3x\}$ and let $\gamma = \{1, x, x^2, x^3\}$ be the standard ordered basis for $P_3(\mathbb{R})$. We have the coordinate vectors of β in γ as:

$$[1]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [x]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad [-1 + 2x^2]_\gamma = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad [-3x + 4x^3]_\gamma = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 4 \end{pmatrix}$$

Note that the matrix with the coordinate vectors as columns have four pivots

$$Q = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Then $\{[1]_\gamma, [x]_\gamma, [-1 + 2x^2]_\gamma, [-3x + 4x^3]_\gamma\}$ is linearly independent. By Thorem 2.21, β is linearly independent. Combined with the fact that $|\beta| = \dim(P_3(\mathbb{R})) = 4$, β is a basis for $P_3(\mathbb{R})$.

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Problem 3.

Let $\beta = \{e_1, e_2, e_3\}$ be the standard ordered basis for R^3. The matrix representation of T in β is

$$[T]_\beta = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note that the augmented matrix

$$[[T]_\beta \,|\, I_3] = \begin{pmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2/3 & 1 & 0 \end{pmatrix} = \left[I_3 \,|\, ([T]_\beta)^{-1}\right]$$

So we have

$$[T]_\beta^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 1 \\ 2/3 & 1 & 0 \end{pmatrix}$$

By Theorem 2.18, T is invertible and $[T^{-1}]_\beta = ([T]_\beta)^{-1}$. We have

$$T^{-1}(a, b, c) = (a/3, c, 2a/3 + b).$$

Problem 4.

(a) Note that, in the matrix and column vector notation, we have

$$x \in \mathbb{R}^3 \mapsto T(x) = Ax, \quad A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then, T is linear.

(b) To find the null space of T, row reduce the augmented matrix corresponding to $Ax = 0$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Then

$$\text{the null space of } T = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

The range of T is the column space of A and we have

$$\text{the range of } T = \text{span}\,\{\text{pivot columns of } A\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

(c) The nullity of T is 1 and the rank of T is 2 We have

$$\text{nullity}(T) + \text{rank}(T) = 1 + 2 = 3 = \dim(\mathbb{R}^3).$$

Then, the dimension theorem is verified.

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Problem 5.

(a) $W_1 + W_2$ is a subspace of W: Closed under vector addition, because if u, $v \in W_1 + W_2$, then there exist u_1, $v_1 \in W_1$ and u_2, $v_2 \in W_2$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$, and then $u + v = u_1 + u_2 + v_1 + v_2 = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2$. For scalar multiplication, $au = a(u_1 + u_2) = au_1 + au_2 \in W_1 + W_2$. Finally, $W_1 + W_2$ contains 0 since both W_1, W_2 are subspaces and therefore contain 0. $W_1 + W_2$ contains both W_1 and W_2: Every vector in $W_1 + W_2$ has the form $x + y$ with $x \in W_1$, $y \in W_2$. Set $y = 0$ to obtain all vectors in W_1 and $x = 0$ to obtain all vectors in W_2. That is, any vector $x \in W_1$ or $y \in W_2$ is also present in $W_1 + W_2$.

(b) A subspace W of V that contains both W_1 and W_2 must also contain all vectors of the form $x + y$ with $x \in W_1$, $y \in W_2$, since it is closed under addition. Therefore it contains $W_1 + W_2$.

(c) ($\Rightarrow$) $W_1 + W_2 \subseteq \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$: For any $\mathbf{u} \in W_1 = \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ and $\mathbf{v} \in W_2 = \text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_q\}$, there exist $c_1, \cdots, c_p$ and $d_1, \cdots, d_q$ such that

$$\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p, \quad \mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q.$$

Then

$$\mathbf{u} + \mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p + d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q \in \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$$

($\Leftarrow$) $\text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\} \subseteq W_1 + W_2$: For any $\mathbf{w} \in \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$, there exist $c_1, \cdots, c_p$ and $d_1, \cdots, d_q$ such that

$$\mathbf{w} = (c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) + (d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q) \in W_1 + W_2$$

Problem 6. (BONUS PROBLEM)

(a) (1) $T(V_0)$ contains 0_W, since $0_V \in V_0$ and $T(0_V) = 0_W$. (2) Let u_1, $u_2 \in T(V_0)$, then there exist v_1, $v_2 \in V_0$ such that $T(v_1) = u_1$ and $T(v_2) = u_2$. Then $v_1 + v_2 \in V_0$, and $T(V_0) \ni T(v_1 + v_2) = T(v_1) + T(v_2) = u_1 + u_2$. (3) Similarly for scalar multiplication, let $u \in T(V_0)$, then there exists $v \in V_0$ such that $T(v) = u$. Then $av \in V_0$, and $T(V_0) \ni T(av) = aT(v) = au$. Combining (1)-(3) shows that $T(V_0)$ is a subspace of W.

(a') Let $\beta = \{u_1, \cdots, u_n\}$ be a basis for V_0. T being linear, we have

$$T(V_0) = \{T(v), \forall v \in V_0\} = \{T(a_1u_1 + \cdots + a_nu_n), \forall a_1, \cdots, a_n \in F\}$$
$$= \{a_1T(u_1) + \cdots + a_nT(u_n), \forall a_1, \cdots, a_n \in F\} = \text{span}\{T(u_1), \cdots, T(u_n)\}$$

Then $T(V_0)$ is a subspace of W.

(b) Let $\beta = \{u_1, \cdots, u_n\}$ be a basis for V_0. $T(\beta)$ is then a basis for $T(V_0)$ (from (a')), since it spans $T(V_0)$ and its vectors are linearly independent:

$$a_1T(u_1) + \cdots + a_nT(u_n) = T(a_1u_1 + \cdots + a_nu_n) = 0$$

gives $a_1u_1 + \cdots + a_nu_n = 0$ since T is an isomorphism, and $a_1 = \cdots = a_n = 0$ since β is a basis for V_0. Thus, $n = \dim(V_0) = \dim(T(V_0))$.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.