1. Label the following statements are true or false

(a) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.
(b) An \( n \times n \) matrix having rank \( n \) is invertible.
(c) Any homogeneous system of linear equations has at least one solution.
(d) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.
(e) The determinant of an upper triangular \( n \times n \) matrix is the product of its diagonal entries.
(f) A matrix \( A \in M_{n \times n}(F) \) has rank \( n \) if and only if \( \det(A) \neq 0 \).
(g) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.
(h) Similar matrices always have the same eigenvalues.
(i) Let \( A \in M_{n \times n}(F) \) and \( \beta = \{v_1, v_2, \ldots, v_n\} \) be an ordered basis for \( F^n \) consisting of eigenvectors of \( A \). If \( Q \) is the \( n \times n \) matrix whose \( j \)th column is \( v_j \) \((1 \leq j \leq n)\), then \( Q^{-1}AQ \) is a diagonal matrix.
(j) A linear operator \( T \) on a finite-dimensional vector space is diagonalizable if and only if the characteristic polynomial of \( T \) splits and the multiplicity of each eigenvalue \( \lambda \) equals the dimension of \( E_\lambda \).

2. Describe the solution set of

\[
\begin{align*}
2x_1 + 4x_2 - 6x_3 &= 0, \\
nx_1 + 8x_2 - 10x_3 &= 0;
\end{align*}
\]

compare it to the solution set

\[
\begin{align*}
2x_1 + 4x_2 - 6x_3 &= 0, \\
nx_1 + 8x_2 - 10x_3 &= 4.
\end{align*}
\]

3. Prove that if \( B \) is a \( n \times 1 \) matrix and \( C \) is a \( 1 \times n \) matrix, then the \( n \times n \) matrix \( BC \) has rank at most 1. Conversely, show that if \( A \) is any \( n \times n \) matrix having rank 1, then there exists a \( n \times 1 \) matrix \( B \) and a \( 1 \times n \) matrix \( C \) such that \( A = BC \).

4. Compute the determinant of each of the following matrices

(a) \[
\begin{pmatrix}
1 & 0 & 2 & 3 \\
4 & 2 & 1 & 3 \\
0 & 0 & 3 & 4 \\
0 & 0 & 4 & 5
\end{pmatrix},
\]
(b) \[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 2 & 3 & 4 \\
0 & 0 & 3 & 4 \\
0 & 0 & 0 & 4
\end{pmatrix},
\]
(c) \[
\begin{pmatrix}
0 & 0 & 2 & 0 \\
0 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}.
\]

5. Let \( A \in M_{n \times n}(F) \) such that \( A^k = 0 \) for some positive integer \( k \). Prove that \( A \) is not invertible.

6. For \( A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \), find an expression for \( e^A \). (Recall that the exponential of \( A \), denoted by \( e^A \), is the matrix given by the power series \( e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \).)
7. Prove that similar matrices have the same characteristic polynomial.

8. (BONUS PROBLEM) Let $A \in M_{n \times n}(F)$. Prove that if $A$ is diagonalizable, then $A^t$ is also diagonalizable.
Problem 1.

(a) True. From Corollary 2 in Page 158, \( \text{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A) \).

(b) True. From Theorem 2.5, \( L_A : F^n \to F^n \) is invertible if and only if \( \text{rank}(L_A) = \dim(F^n) \), i.e., \( \text{rank}(A) = n \). From Corollary 2 in Page 102, \( A \) is invertible if and only if \( L_A \) is invertible. Then \( A \) is invertible if and only if \( \text{rank}(A) = n \).

(c) True. Any homogeneous system of linear equations has zero as a solution.

(d) False. For example, the system that \( 0x = 1 \) has no solution while the corresponding homogeneous system \( 0x = 0 \) has a solution.

(e) True. It is from Problem 4.2.23 in Page 222.

(f) True. From Property 7 in Page 236, \( A \) is invertible if and only if \( \det(A) \neq 0 \). Combined with the result (b) above, \( A \) has rank \( n \) if and only if \( \det(A) \neq 0 \).

(g) True. If \( v \in E_{\lambda} \setminus \{0\} \), i.e., \( v \) is an eigenvector of \( A \), then \( cv \in E_{\lambda} \setminus \{0\} \) for any \( c \in \mathbb{R} \setminus \{0\} \).

(h) True. From Problem 5.1.12(a), similar matrices have the same characteristic polynomial, thus have the same eigenvalues.

(i) True. Let \( Q = (v_1, \ldots, v_n) \) be the \( n \times n \) matrix whose \( j \)th column is \( v_j \) (\( 1 \leq j \leq n \)) such that \( Av_j = \lambda_j v_j \), then \( AQ = A(v_1, \ldots, v_n) = (\lambda_1 v_1, \ldots, \lambda_n v_n) = (v_1, \ldots, v_n) \text{diag}(\lambda_1, \ldots, \lambda_n) = QD \) with \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \). Therefore, \( Q^{-1}AQ = D \) is a diagonal matrix.

(j) True. It is from Theorem 5.9 (i.e., the test for diagonalization).

Problem 2. 

Note that the corresponding augmented matrix to

\[
\begin{align*}
2x_1 + 4x_2 - 6x_3 &= 0, \\
4x_1 + 8x_2 - 10x_3 &= 0.
\end{align*}
\]

is

\[
\begin{pmatrix}
2 & 4 & -6 & 0 \\
4 & 8 & -10 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 & -3 & 0 \\
0 & 0 & 2 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

The vector form of the solution is

\[
x = \begin{pmatrix}
-2x_2 \\
x_2 \\
0
\end{pmatrix}
= x_2 \begin{pmatrix}
-2 \\
1 \\
0
\end{pmatrix}.
\]

The corresponding augmented matrix to

\[
\begin{align*}
2x_1 + 4x_2 - 6x_3 &= 0, \\
4x_1 + 8x_2 - 10x_3 &= 4.
\end{align*}
\]

is

\[
\begin{pmatrix}
2 & 4 & -6 & 0 \\
4 & 8 & -10 & 4
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 & 0 & 6 \\
0 & 0 & 1 & 2
\end{pmatrix}.
\]

The vector form of the solution is

\[
x = \begin{pmatrix}
6 \\
0 \\
2
\end{pmatrix} + x_2 \begin{pmatrix}
-2 \\
1 \\
0
\end{pmatrix}.
\]
Problem 3.

(⇒) If $B = b \in F^{n \times 1}$ and $C = c^T = (c_1, \cdots, c_n) \in F^{1 \times n}$, then

$$BC = bc^T = b(c_1, \cdots, c_n) = (c_1b, \cdots, c_n b),$$

thus $\text{range}(BC) = \text{span}\{b\}$ and $\text{rank}(BC) = \text{dim}(\text{span}\{b\}) \leq 1$.

(⇐) If $A = (a_1, \cdots, a_n) \in F^{n \times n}$ have rank 1, then

$$\text{range}(A) = \text{span}\{a_1, \cdots, a_n\} = \text{span}\{b\}$$

for some $b \neq 0 \in F^{n \times 1}$, and $\exists c^T = (c_1, \cdots, c_n) \in F^{1 \times n}$ such that

$$a_1 = c_1 b, \cdots, a_n = c_n b.$$

Therefore,

$$A = (c_1 b, \cdots, c_n b) = b(c_1, \cdots, c_n) = BC$$

with $B = b$ and $C = c^T$.

Problem 4.

(a)

$$\text{det} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{pmatrix} = \text{det} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{det} \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} = (1 \cdot 2)(3 \cdot 5 - 4 \cdot 4) = -2.$$

(b)

$$\text{det} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$

(c)

$$\text{det} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = -\text{det} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = -(3 \cdot 1 \cdot 2 \cdot 4) = -24.$$
Name and ID: ____________________________

**Problem 5.**

Proof. By the fact that $\det(BC) = \det(B)\det(C)$ and an easy induction argument, we have

$$0 = \det(0) = \det(A^k) = \prod_{i=1}^{k} \det(A) = (\det(A))^k.$$ 

Taking $k$th roots, we have $\det(A) = 0$, so $A$ is not invertible.

**Problem 6.**

Note that $A$ has two distinct eigenvalues $5$ and $-1$, thus is diagonalizable, i.e.,

$$Q^{-1}AQ = D \quad \text{with} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad Q^{-1} = \frac{1}{3}Q.$$ 

Note that

$$A^k = QD^kQ^{-1}, \quad e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = Q \left( \sum_{k=0}^{\infty} \frac{1}{k!} D^k \right) Q^{-1} = Qe^DQ^{-1}$$ 

So we have

$$e^A = Qe^DQ^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^5 & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} e^5 + 2e^{-1} & 2e^5 - 2e^{-1} \\ e^5 - e^{-1} & 2e^5 + e^{-1} \end{pmatrix}.$$
Problem 7.
Let $A$ and $B$ be similar, i.e., $\exists Q$ invertible such that $B = Q^{-1}AQ$. Note that $\det(Q^{-1}) = (\det(Q))^{-1}$. We have

$$p_B(\lambda) = \det(B - \lambda I_n) = \det(Q^{-1}AQ - \lambda I_n) = \det(Q^{-1}AQ - \lambda Q^{-1}I_n Q) = \det(Q^{-1}) \det(A - \lambda I_n) \det(Q) = \det(A - \lambda I_n) \det(Q^{-1})^{-1} \det(Q) = \det(A - \lambda I_n) = p_A(\lambda).$$

Problem 8. (BONUS PROBLEM)
(1st proof:)
Note that $(BC)^t = C^t B^t$ for any dimension consistent matrices $B$ and $C$. If $A$ is diagonalizable, then there are an invertible matrix $Q$ and a diagonal matrix $D$ such that $D = Q^{-1}AQ$.
Taking the transpose on both sides gives

$$D = D^t = (Q^{-1}AQ)^t = (Q)^t (A^t)(Q^{-1})^t = \tilde{Q}^{-1}A^t\tilde{Q}$$
with $\tilde{Q} = (Q^{-1})^t$ and we have $\tilde{Q}^{-1} = Q^t$. Therefore, $A^t$ is diagonalizable.

(2nd proof:)
If $A$ is diagonalizable, then from Theorem 5.6, the characteristic polynomial of $A$ splits. Let $m_\lambda$ be the multiplicity of $\lambda$ as an eigenvalue of $A$. From Theorem 5.9, we have $\dim(E_\lambda) = m_\lambda$. Note that $A$ and $A^t$ have the characteristic polynomial. Then $m_\lambda$ is also the multiplicity of $\lambda$ as an eigenvalue of $A^t$. Note that $E_\lambda = N(A - \lambda I)$ and $E'_{\lambda} = N(A^t - \lambda I)$. By the dimension theorem (Theorem 2.3), we have

$$\dim(E_\lambda) = \dim(N(A - \lambda I)) = n - \text{rank}(A - \lambda I) = n - \text{rank}((A - \lambda I)^t)$$
$$= n - \text{rank}(A^t - \lambda I) = \dim(N(A^t - \lambda I)) = \dim(E'_{\lambda})$$

since $\text{rank}B^t = \text{rank}B$ for any matrix $B$. Therefore, we have $\dim(E'_{\lambda}) = \dim(E_\lambda) = m_\lambda$. From Theorem 5.9, $A^t$ is diagonalizable.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.