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10 points 1. Mark each statement True or False. Justify each answer.

(1) A vector is an arrow in three-dimensional space.

(2) A subset H of a vector space V is a subspace of V if the zero vector is in H.

(3) A subspace is also a vector space.

(4) If $\mathbf{u}$ is a vector in a vector space V, then $(-1)\mathbf{u}$ is the same as the negative of $\mathbf{u}$.

(5) A vector space is also a subspace.

(6) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

(7) If f is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $f(t) = 0$ for some t, then f is the zero vector in V.

(8) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

(9) Any set containing the zero vector is linearly dependent.

(10) Subsets of linearly dependent sets are linearly dependent.

15 points 2. Let W be the set of all vectors of the form $\begin{pmatrix} a+2b+4c \\ b+2c \\ -a+3b+6c \end{pmatrix}$ where a, b and c are arbitrary.

a. Find vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ such that $W = \mathbf{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

b. Find a basis for W and state the dimension of W.

15 points 3. Suppose $\mathbf{u}_1, \cdots, \mathbf{u}_p$ and $\mathbf{v}_1, \cdots, \mathbf{v}_q$ are vectors in a vector space V, and let

$$H = \mathbf{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}, \quad K = \mathbf{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_q\}.$$

Show that $H + K = \mathbf{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$.

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Problem 1.

(1) False. A vector is an element of a vector space.

(2) False. A subset H of a vector space V is a subspace of V if (a) the zero vector is in H, (b) H is closed under vector addition, and (c) H is closed under scalar multiplication.

(3) True. A subspace is also a vector space in its own right.

(4) True. If $\mathbf{u}$ is a vector in a vector space V, then, by Theorem 1.2, $(-1)\mathbf{u} = -(1\mathbf{u}) = -\mathbf{u}$.

(5) True. A vector space is also a subspace of itself.

(6) False. $\mathbb{R}^2$ is a vector space, but is not a subset of $\mathbb{R}^3$.

(7) False. The function f_0 such that $f_0(t) = 0$ for ANY t is the UNIQUE zero vector f_0 in V.

(8) Fasle. $S = \{v, 0\}$ with $v \neq 0$ is linear dependent, but v is not a linear combination of 0. If S is a linearly dependent set of two or more vectors, then AT LEAST ONE of the vectors in S is a linear combination of other vectors in S.

(9) True. Refer to lecture notes for Section 1.5.

(10) Fasle. $S = \{v, 0\}$ with $v \neq 0$ is linear dependent, but subset $\{v\}$ is not. Refer to Theorem 1.6 in Section 1.5.

Problem 2.

a. Note that

$$\begin{pmatrix} a+2b+4c \\ b+2c \\ -a+3b+6c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c\begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

We have

$$W = \left\{ \begin{pmatrix} a+2b+4c \\ b+2c \\ -a+3b+6c \end{pmatrix}, \quad a, b, c \in \mathbb{R} \right\} = \mathbf{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\},$$

where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}.$$

W is a subspace of $\mathbb{R}^3$ since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are in $\mathbb{R}^3$, and by Theorem 1.5, $\mathbf{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a subspace of $\mathbb{R}^3$

b. By row reduction,

$$(\mathbf{u}_1\, \mathbf{u}_2, \mathbf{u}_3) = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The pivot columns $\{\mathbf{u}_1, \mathbf{u}_2\}$ form a basis for W and $\dim(W) = 2$.

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Problem 3.

(a) $H + K \subseteq \mathbf{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$: For any $\mathbf{u} \in H = \mathbf{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ and $\mathbf{v} \in K = \mathbf{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_q\}$, there exist $c_1, \cdots, c_p$ and $d_1, \cdots, d_q$ such that

$$\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p, \quad \mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q.$$

Then

$$\mathbf{u} + \mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p + d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q \in \mathbf{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$$

(b) $\mathbf{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\} \subseteq H + K$: For any $\mathbf{w} \in \mathbf{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_p, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$, we have

$$\mathbf{w} = (c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) + (d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q) \in H + K$$

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.