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15 points 1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case). Let $v_1, \cdots, v_p$ be vectors in a non-zero finite-dimensional vector space V, and $S = \{v_1, \cdots, v_p\}$.

(1) The set of all linear combinations of $v_1, \cdots, v_p$ is a vector space.

(2) If $\{v_1, \cdots, v_{p-1}\}$ spans V, then S spans V.

(3) If $\{v_1, \cdots, v_{p-1}\}$ is linearly independent, then so is S.

(4) If S is linearly independent, then S is a basis for V.

(5) If S is linearly independent, then $\dim V \geq p$.

(6) If $V = \text{span}\{S\}$, then some subset of S is a basis for V.

(7) If $V = \text{span}\{S\}$, then $\dim V \leq p$.

(8) If $\dim V = p$ and $V = \text{span}\{S\}$, then S cannot be linearly dependent.

(9) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

(10) If A is $m \times n$ and $\text{rank}A = m$, then the linear transformation $x \mapsto Ax$ is one-to-one.

15 points 2. Let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -10 \end{pmatrix}, \quad c = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x) = Ax$ for every $x \in \mathbb{R}^3$.

(1) Find an x in $\mathbb{R}^3$ whose image under T is b.

(2) Is there more than one x under T whose image is b.

(3) Determine if c is in the range of the transformation T.

(4) Find the null space and the range of T.

(5) Find the nullity and rank of T and verify the dimension theorem.

10 points 3. Use coordinate vectors to determine if

$$\beta = \{1 - t, 2 - t + t^2, 2t + 3t^2\}$$

is a basis for $P_2(\mathbb{R})$.

10 points 4. (**BONUS PROBLEM**) Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\{v_1, \cdots, v_p\}$ be a subset of V. Suppose that T is one-to-one transformation, so that $T(u) = T(v)$ always implies $u = v$. Show that if $\{v_1, \cdots, v_p\}$ is linearly independent in V, then the set $\{T(v_1), \cdots, T(v_p)\}$ is linearly independent in W.

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Problem 1. Problem 1.

(1) True. $\text{span}\{S\} = \{\text{all linear combinations of } v_1, \cdots, v_p\}$ is a subspace of V.

(2) True. $\text{span}\{v_1, \cdots, v_{p-1}\} \subseteq \text{span}\{S\} \subseteq V$. If $\text{span}\{v_1, \cdots, v_{p-1}\} = V$, then $\text{span}\{S\} = V$.

(3) False. If $\{v_1, \cdots, v_{p-1}\}$ is linearly independent, then so is S only provided that $v_p \notin \text{span}\{v_1, \cdots, v_{p-1}\}$.

(4) False. If S is linearly independent, then S is a basis for V only provided that $V = \text{span}\{S\}$.

(5) True. If S is linearly independent, then, by Theorem 1.11, $\dim V \geq p$.

(6) True. If $V = \text{span}\{S\}$, then, by Theorem 1.9, some subset of S is a basis for V.

(7) True. If $V = \text{span}\{S\}$, then, by Theorem 1.9, $\dim V \leq p$.

(8) True. If $\dim V = p$ and $V = \text{span}\{S\}$, then, by Theorem 1.10, Corollary 2, S is a basis for V, therefore, S cannot be linearly dependent.

(9) False. A plane in $\mathbb{R}^3$ that PASSES THROUGH the origine is a two-dimensional subspace.

(10) False. If A is $m \times n$ and $\text{rank}A = m$, then $\text{range}(T) = \mathbb{R}^m$, $m \leq n$, and the linear transformation $x \in \mathbb{R}^n \mapsto Ax \in \mathbb{R}^m$ is onto. T is one-to-one if we also have $m = n$ since, by Theorem 2.3, $\text{nullity}(T) = n - \text{rank}(T) = n - m = 0$.

Problem 2.

(1) To find x whose image under T is b, solve $Ax = b$ or

$$\begin{pmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}$$

Note that the augmented matrix

$$\begin{pmatrix} 1 & -2 & 3 & 2 \\ -5 & 10 & -15 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have

$$x_1 = 2x_2 - 3x_3 + 2, \qquad x_2 \text{ and } x_3 \text{ are free.}$$

Let $x_2 = x_3 = 0$. Then $x_1 = 2$. We have

$$x = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

whose image under T is b

(2) Since free variables exist, there is more that one x for which $T(x) = b$.

(3) To determine if c is in the range of the transformation T, we consider whether $Ax = c$ is consistent. Note that the augmented matrix

$$\begin{pmatrix} 1 & -2 & 3 & 3 \\ -5 & 10 & -15 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then c is not in the range of T.

(4) To find the null space of T, row reduce the augmented matrix corresponding to $Ax = 0$

$$\begin{pmatrix} 1 & -2 & 3 & 0 \\ -5 & 10 & -15 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Then

$$\text{the null space of } T = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The range of T is the column space of A and we have

$$\text{the range of } T = \text{span}\,\{\text{pivot columns of } A\} = \text{span}\left\{ \begin{pmatrix} 1 \\ -5 \end{pmatrix} \right\}$$

(5) The nullity of T is 2 and the rank of T is 1, we have

$$\text{nullity}(T) + \text{rank}(T) = \dim(\mathbb{R}^3) = 3.$$

Then, the dimension theorem is verified.

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Problem 3.

Let $\gamma = \{1, x, x^2\}$ be the standard ordered basis for $P_2(\mathbb{R})$. We have the coordinate vectors of $\beta = \{1-t, 2-t+t^2, 2t+3t^2\}$ in γ as:

$$[1-t]_\gamma = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad [2-t+t^2]_\gamma = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad [2t+3t^2]_\gamma = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Row reduce the matrix with the coordinate vectors as columns

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

By the inverse matrix theorem, $\{[1-t]_\gamma, [2-t+t^2]_\gamma, [2t+3t^2]_\gamma\}$ is linearly independent. By Thorem 2.21, β is linearly independent. Combined with the fact that $|\beta| = \dim(P_2(\mathbb{R})) = 3$, β is a basis for $P_2(\mathbb{R})$.

Problem 4. (BONUS PROBLEM)

Let $c_1, \cdots, c_p$ such that

$$c_1T(v_1) + \cdots + c_pT(v_p) = 0. \tag{1}$$

T being linear implies from (1) that

$$c_1T(v_1) + \cdots + c_pT(v_p) = T(c_1v_1 + \cdots + c_pv_p) = 0. \tag{2}$$

T being one-to-one implies from (2) that

$$c_1v_1 + \cdots + c_pv_p = 0 \tag{3}$$

$\{v_1, \cdots, v_p\}$ being linearly independent implies from (3) that

$$c_1 = \cdots = c_p. \tag{4}$$

Therefore, from (1) and (4), $\{T(v_1), \cdots, T(v_p)\}$ is linearly independent.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.